FINITE CONFORMAL MODULES OVER $N = 2, 3, 4$ SUPERCONFORMAL ALGEBRAS

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Abstract. In this paper we continue the study of representation theory of formal distribution Lie superalgebras initiated in [4]. We study finite Verma-type conformal modules over the $N = 2$, $N = 3$ and the two $N = 4$ superconformal algebras and also find explicitly all singular vectors in these modules. From our analysis of these modules we obtain a complete list of finite irreducible conformal modules over the $N = 2$, $N = 3$ and the two $N = 4$ superconformal algebras.

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1. Introduction

Superconformal algebras have been playing an important role in the study of string theory and conformal field theory, which have been the subject of intensive study since the seminal paper [2]. Superconformal algebras may be viewed as natural super-extensions of the Virasoro algebra and their roots in physics literature can be traced at least back to as early as the 70's [1]. A mathematically rigorous definition of a superconformal algebra is as follows. It is a simple Lie superalgebra $\mathfrak{g}$ over the complex numbers $\mathbb{C}$ spanned by the modes of a finite family $\mathfrak{F}$ of mutually local fields satisfying the following two axioms [7]:

1. $\mathfrak{F}$ contains the Virasoro field,
2. the coefficients of the operator product expansions of members from $\mathfrak{F}$ are linear combinations of members from $\mathfrak{F}$ and their derivatives.

A Lie superalgebra $\mathfrak{g}$ satisfying the second axiom only is referred to as a *formal distribution Lie superalgebra* in [7].

In order to facilitate the study of formal distribution Lie superalgebras the notion of a *conformal superalgebra* was introduced in [7] (see Section 2). It proves to be an effective tool for this purpose.

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A natural class of representations of formal distribution Lie superalgebras to study is the class of *conformal modules* [4]. A conformal module is a pair consisting of a $\mathfrak{g}$-module V and a family $\mathcal{E}$ of fields whose modes span V such that members from $\mathfrak{F}$ and $\mathcal{E}$ are mutually local. Just as the study of formal distribution Lie superalgebras reduces to the study of conformal superalgebras, the study of conformal modules is essentially reduced to the study of modules over the corresponding conformal superalgebras.

The study of modules over the conformal superalgebra can further be reduced to the study of modules over the *extended annihilation subalgebra*, which is a semidirect sum of the subalgebra of positive modes of the corresponding formal distribution Lie superalgebra and a one-dimensional derivation. It is in this language that the problem of classifying finite irreducible conformal modules over the Virasoro, $N = 1$ (Neveu-Schwarz) and the current superalgebra was solved in [4].

The problem of classifying conformal modules over other superconformal algebras, which is the main theme of the present paper, turns out to be more subtle. The main purpose here is to give a classification of finite irreducible conformal modules over the $N = 2$, $N = 3$ and the two $N = 4$ superconformal algebras.

We first construct finite Verma-type conformal modules for a general superconformal algebra and prove that every finite irreducible conformal module is a homomorphic image of such a module. As a consequence we obtain a bijection between finite irreducible conformal modules of a superconformal algebra and finite-dimensional irreducible modules of a certain finite-dimensional reductive Lie (super)algebra (Corollary 3.1).

We then study these Verma-type modules in detail for the four members of the family of superconformal algebras mentioned above. It turns out that, unlike for the Virasoro and the $N = 1$ (Neveu-Schwarz) superconformal algebras, the Verma-type modules for these superconformal algebras are in general reducible, and thus we need to analyze their submodules. This is accomplished by finding explicit formulas for all singular vectors inside such a module and then show that the submodule generated by these singular vectors is maximal (in all but two cases). We also find an explicit basis for this maximal submodule, which then enables us to give a quite explicit description of all finite irreducible conformal modules over these superconformal algebras.

This paper is organized as follows. In Section 2 basic facts of formal distribution Lie superalgebras, conformal superalgebras and extended annihilation subalgebras are recalled. Section 3 is devoted to the study of a class of modules over a certain class of Lie superalgebras that include the annihilation subalgebra of every superconformal algebra. This class of modules gives rise to finite Verma-type conformal modules of superconformal algebras. The results of Section 3 are then used in Section 4, Section 5, Section 6 and Section 7, where finite irreducible

conformal modules over the $N = 2$, $N = 3$, the "small" $N = 4$ and the "big" $N = 4$ superconformal algebra, respectively, are classified.

In this paper all vector spaces, (super)algebras and tensor products are over taken over the complex numbers $\mathbb{C}$.

2. Preliminaries

In this section we review some of the basic facts on formal distribution Lie (super)algebras and conformal modules that will be used later on. The material here is taken from [4], [7] and [9], and the reader is referred to these articles for more details.

2.1. **Formal Distribution Lie Superalgebras.** Recall that a *formal distribution* or a *field* with coefficients in a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is a formal series of the form:

$$a(z) = \sum_{n\in\mathbb{Z}} a_{[n]} z^{-n-1},$$

where $a_{[n]} \in \mathfrak{g}$ and z is an indeterminate.

Two formal distributions $a(z)$ and $b(z)$ with coefficients in $\mathfrak{g}$ are said to be mutually *local* if there exists $N \in \mathbb{Z}_+$ such that

$$(z-w)^N [a(z), b(w)] = 0. \tag{2.1}$$

Let $\delta(z-w) = z^{-1}\sum_{n\in\mathbb{Z}}(\frac{z}{w})^n$ be the formal delta function. Then (2.1) may be written as

$$[a(z), b(w)] = \sum_{j=0}^{N-1} (a_{(j)}b)(w)\partial_w^{(j)}\delta(z-w), \tag{2.2}$$

(here $\partial_w^{(j)}$ stands for $\frac{1}{j!}\frac{\partial^j}{\partial w^j}$) for some uniquely determined formal distributions $(a_{(j)}b)(w)$, and thus defines a $\mathbb{C}$-bilinear product $\cdot_{(j)}\cdot$ for each $j \in \mathbb{Z}_+$ on the space of all formal distributions with coefficients in $\mathfrak{g}$. Also $\partial_z a(z) = \sum_n (\partial a)_{[n]} z^{-n-1}$, where $(\partial a)_{[n]} = -na_{[n-1]}$, and hence the space of all formal distributions is also a (left) $\mathbb{C}[\partial_z]$-module.

A Lie superalgebra $\mathfrak{g}$ is called a *formal distribution Lie superalgebra*, if there exists a family $\mathfrak{F}$ of mutually local formal distributions whose coefficients span $\mathfrak{g}$. We will write $(\mathfrak{g}, \mathfrak{F})$ for such a Lie superalgebra.

Given a formal distribution Lie superalgebra $(\mathfrak{g}, \mathfrak{F})$, we may include $\mathfrak{F}$ in the minimal family $\overline{\mathfrak{F}}$ of mutally local distributions which is closed under ∂_z and all products $\cdot_{(j)}\cdot$. Then $\overline{\mathfrak{F}}$ is a *conformal superalgebra*, i.e. it is a left $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module R with a $\mathbb{C}$-bilinear product $a_{(n)}b$ for each $n \in \mathbb{Z}_+$ such that the following axioms hold ($a, b, c \in R; m, n \in \mathbb{Z}_+$ and $\partial^{(j)} = \frac{1}{j!}\partial^j$) (cf. [3], [6]):

(C0) $a_{(n)}b = 0$, for $n >> 0$,
(C1) $(\partial a)_{(n)}b = -na_{(n-1)}b$,

(C2) $a_{(n)}b = (-1)^{p(a)p(b)} \sum_{j=0}^{\infty} (-1)^{j+n+1} \partial^{(j)}(b_{(n+j)}a)$,
(C3) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c + (-1)^{p(a)p(b)} b_{(n)}(a_{(m)}c)$.

It is convenient to write the products of $a, b \in R$ in the generating series form

$$a_\lambda b = \sum_{n=0}^{\infty} a_{(n)} b \frac{\lambda^n}{n!},$$

where λ is a formal indeterminate. Such an expression lies in $R[\lambda]$.

Conversely, if a conformal superalgebra $R = \bigoplus_{i \in I} \mathbb{C}[\partial]a^i$ is free $\mathbb{C}[\partial]$-module, we may associate to R a formal distribution Lie superalgebra $(\mathfrak{g}(R), \mathfrak{F}(R))$ with Lie superalgebra $\mathfrak{g}(R)$ spanned by $\mathbb{C}$-basis $a^i_{[m]}$ ($i \in I$, $m \in \mathbb{Z}$) and fields $\mathfrak{F}(R) = \{a^i(z) = \sum_{n \in \mathbb{Z}} a^i_{[n]} z^{-n-1}\}_{i \in I}$ with bracket (cf. (2.2)):

$$[a^i(z), a^j(w)] = \sum_{k \in \mathbb{Z}_+} (a^i_{(k)} a^j)(w) \partial_w^{(k)} \delta(z-w),$$

so that $\overline{\mathfrak{F}(R)} = R$, giving rise to commutation relations ($m, n \in \mathbb{Z}$; $i, j \in I$)

$$[a^i_{[m]}, a^j_{[n]}] = \sum_{k \in \mathbb{Z}_+} \binom{m}{k} (a^i_{(k)} a^j)_{[m+n-k]}. \tag{2.3}$$

It follows that the Lie superalgebra $\mathfrak{g}$ of a formal distribution Lie superalgebra $(\mathfrak{g}, \mathfrak{F})$ is isomorphic to $\mathfrak{g}(\overline{\mathfrak{F}})$ divided by an *irregular* ideal, that is an ideal which does not contain every $a_{[n]}$ for some non-zero element $a \in \overline{\mathfrak{F}}$.

Example 2.1. The (centerless) *Virasoro algebra* $\mathfrak{V}$ has a basis L_n ($n \in \mathbb{Z}$) with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}.$$

It is spanned by the coefficients of the field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfying

$$[L(z), L(w)] = \partial_w L(w) \delta(z-w) + 2L(w) \partial_w \delta(z-w). \tag{2.4}$$

The conformal algebra associated to the Virasoro algebra, is the *Virasoro conformal algebra* $R(\mathfrak{V}) = \mathbb{C}[\partial] \otimes L$ with products $L_\lambda L = (\partial + 2\lambda)L$.

Example 2.2. Let $\mathfrak{g}$ be a finite-dimensional Lie (super)algebra. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ denote the corresponding *current algebra* with bracket

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t), \quad a, b \in \mathfrak{g}; f(t), g(t) \in \mathbb{C}[t, t^{-1}].$$

For each $a \in \mathfrak{g}$ define a field $a(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) z^{-n-1}$. Then $\tilde{\mathfrak{g}}$ is spanned by the coefficients of $a(z)$ satisfying

$$[a(z), b(w)] = [a, b](w) \delta(z-w). \tag{2.5}$$

The conformal (super)algebra associated to the current algebra is the *current conformal algebra* $R(\tilde{\mathfrak{g}}) = \mathbb{C}[\partial] \otimes \mathfrak{g}$ with products $a_\lambda b = [a, b]$, $a, b \in \mathfrak{g}$.

Example 2.3. The semidirect sum $\mathfrak{V} \ltimes \tilde{\mathfrak{g}}$ is another example of a formal distribution Lie (super)algebra. The collection of fields is $\{L(z), a(z) | a \in \mathfrak{g}\}$ and we have in addition to (2.4) and (2.5)

$$[L(z), a(w)] = \partial_w a(w)\delta(z-w) + a(w)\partial_w \delta(z-w). \tag{2.6}$$

The conformal algebra associated to the semidirect sum of the Virasoro algebra and the current algebra is $R(\mathfrak{V} \ltimes \tilde{\mathfrak{g}}) = R(\mathfrak{V}) \ltimes R(\tilde{\mathfrak{g}})$. For $a \in \mathfrak{g}$ we have $L_\lambda a = (\partial + \lambda)a$.

2.2. **Conformal Modules.** Let $(\mathfrak{g}, \mathfrak{F})$ be a formal distribution Lie superalgebra. Let V be a $\mathfrak{g}$-module such that V is spanned over $\mathbb{C}$ by the coefficients of a family $\mathcal{E}$ of fields. If all $a(z) \in \mathfrak{F}$ are local with respect to all $v(z) \in \mathcal{E}$, then the pair $(V, \mathcal{E})$ is called a *conformal module* over $(\mathfrak{g}, \mathfrak{F})$.

Now the family $\mathcal{E}$ of a conformal module $(V, \mathcal{E})$ over $(\mathfrak{g}, \mathfrak{F})$ similarly can be included in a larger family $\overline{\mathcal{E}}$, which is still local with respect to the fields from $\overline{\mathfrak{F}}$, and invariant under ∂ and $a_{(j)}$, for all $a \in \overline{\mathfrak{F}}$ and $j \in \mathbb{Z}_+$. It can be shown that for $a, b \in \overline{\mathfrak{F}}$ and $v \in \overline{\mathcal{E}}$ $(m, n \in \mathbb{Z}_+)$ one has

$$[a_{(m)}, b_{(n)}]v = \sum_{j=0}^{m} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}v, \quad (\partial a)_{(n)}v = [\partial, a_{(n)}]v = -na_{(n-1)}v.$$

Thus it follows that any conformal module $(V, \mathcal{E})$ over a formal distribution Lie superalgebra $(\mathfrak{g}, \mathfrak{F})$ gives rise to a module $M = \overline{\mathcal{E}}$ over the *conformal superalgebra* $R = \overline{\mathfrak{F}}$, defined as follows. It is a (left) $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module equipped with a family of $\mathbb{C}$-linear maps $a \to a^M_{(n)}$ of R to $\mathrm{End}_{\mathbb{C}} M$, for each $n \in \mathbb{Z}_+$, such that the following properties hold for $a, b \in R$ and $m, n \in \mathbb{Z}_+$:

(M0) $a^M_{(n)}v = 0$, for $v \in M$ and $n >> 0$,
(M1) $[a^M_{(m)}, b^M_{(n)}] = \sum_{j=0}^m \binom{m}{j} (a_{(j)}b)^M_{(m+n-j)}$,
(M2) $(\partial a)^M_{(n)} = [\partial, a^M_{(n)}] = -na^M_{(n-1)}$.

Again it is convenient to write the action of an element $a \in R$ on an element $v \in M$ in the form of a generating series in $V[\lambda]$

$$a_\lambda v := \sum_{n=0}^{\infty} a_{(n)}v \frac{\lambda^n}{n!}.$$

Conversely, suppose that a conformal superalgebra $R = \bigoplus_{i \in I} \mathbb{C}[\partial]a^i$ is a free $\mathbb{C}[\partial]$-module and consider the associated formal distribution Lie superalgebra $(\mathfrak{g}(R), \mathfrak{F}(R))$. Let M be a module over the conformal superalgebra R and suppose that M is a free $\mathbb{C}[\partial]$-module with $\mathbb{C}[\partial]$-basis $\{v^\alpha\}_{\alpha \in J}$. This gives rise to a conformal module $V(M)$ over $\mathfrak{g}(R)$ with fields $\mathcal{E} = \{v^\alpha(z) = \sum_{n \in \mathbb{Z}} v^\alpha_{[n]} z^{-n-1} | \alpha \in J\}$

and $\mathbb{C}$-basis $v^{\alpha}_{[n]}$, defined by:

$$a^i(z)v^{\alpha}(w) = \sum_{j\in\mathbb{Z}_+}(a^i_{(j)}v^{\alpha})(w)\partial_w^{(j)}\delta(z-w).$$

A conformal module $(V,\mathcal{E})$ (respectively module M) over a formal distribution Lie superalgebra $(\mathfrak{g},\mathfrak{F})$ (respectively over a conformal superalgebra R) is called *finite*, if $\overline{\mathcal{E}}$ (respectively M) is a finitely generated $\mathbb{C}[\partial]$-module. A conformal module $(V,\mathcal{E})$ over $(\mathfrak{g},\mathfrak{F})$ is called *irreducible*, if there is no non-trivial invariant subspace which contains all $v_{[n]}$, $n\in\mathbb{Z}$, for some non-zero $v\in\overline{\mathcal{E}}$. An invariant subspace that does not contain all $v_{[n]}$, for some non-zero $v\in\mathcal{E}$, is called an *irregular submodule* and conformal modules that differ by an irregular submodule are called referred to as *equivalent* in [9]. Clearly a conformal module is irreducible if and only if the associated module $\overline{\mathcal{E}}$ over the conformal superalgebra $\overline{\mathfrak{F}}$ is irreducible.

Remark 2.1. It follows from (M2) that an eigenvector $v\in M$ of the linear operator ∂ is an R-invariant, i.e. $a_{(n)}v=0$, for all $n\geq 0$. Thus a finite irreducible module over a conformal superalgebra R is either free over $\mathbb{C}[\partial]$ or else it is one-dimensional over $\mathbb{C}$.

Suppose that $(\mathfrak{g},\mathfrak{F})$ is a formal distribution Lie superalgebra such that $\mathfrak{g}(\overline{\mathfrak{F}})\cong\mathfrak{g}$. Our discussion implies that any irreducible conformal module $(V,\mathcal{E})$ over $(\mathfrak{g},\mathfrak{F})$ is a quotient of an irreducible conformal module of the form $V(M)$ divided by an irregular submodule, where M is an irreducible module over the conformal superalgebra $\overline{\mathfrak{F}}$. Hence in particular if $V(M)$ is irreducible as a $\mathfrak{g}$-module for every irreducible M, then every finite irreducible conformal modules over $(\mathfrak{g},\mathfrak{F})$ isomorphic to $V(M)$, for some finite irreducible $\overline{\mathfrak{F}}$-module M.

Example 2.4. The Virasoro algebra $\mathfrak{V}$ may be identified with the Lie algebra of regular vector fields on $\mathbb{C}^{\times}$, where $L_n=-t^{n+1}\frac{d}{dt}$, $n\in\mathbb{Z}$. For $\alpha,\Delta\in\mathbb{C}$ let

$$F_{\mathfrak{V}}(\alpha,\Delta)=\mathbb{C}[t,t^{-1}]e^{-\alpha t}dt^{1-\Delta}.$$

The Lie algebra $\mathfrak{V}$ acts on the space $F_{\mathfrak{V}}(\alpha,\Delta)$ in a natural way:

$$(f(t)\frac{\partial}{\partial t})g(t)dt^{1-\Delta}=(f(t)g'(t)+(1-\Delta)g(t)f'(t))dt^{1-\Delta},$$

where $f(t)\in\mathbb{C}[t,t^{-1}]$ and $g(t)\in\mathbb{C}[t,t^{-1}]e^{-\alpha t}$. Letting $v_{[n]}=t^ne^{-\alpha t}dt^{1-\Delta}$ and $v(z)=\sum_{n\in\mathbb{Z}}v_{[n]}z^{-n-1}$ this action is equivalent to

$$L(z)v(w)=(\partial_w+\alpha)v(w)\delta(z-w)+\Delta v(w)\partial_w\delta(z-w).$$

Hence we have constructed a two-parameter family of conformal modules over $\mathfrak{V}$. This gives a family of $R(\mathfrak{V})$-modules $\mathbb{C}[\partial]\otimes\mathbb{C}v_{\Delta}$ with products $L_{\lambda}v_{\Delta}=(\alpha+\partial+\Delta\lambda)v_{\Delta}$. This module is irreducible if and only if $\Delta\neq 0$, in which case

it will be denoted by $L_{\mathfrak{V}}(\alpha, \Delta)$. We set $L_{\mathfrak{V}}(\alpha, 0)$ to be the one-dimensional (over $\mathbb{C}$) $R(\mathfrak{V})$-module on which ∂ acts as the scalar α.

Example 2.5. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and U^Λ the finite-dimensional irreducible module of highest weight Λ. Then $F_{\tilde{\mathfrak{g}}}(\Lambda) = U^\Lambda \otimes \mathbb{C}[t, t^{-1}]$ is naturally a module over $\tilde{\mathfrak{g}}$ with action given by

(2.7)
$$(a \otimes f(t))(u \otimes g(t)) = au \otimes f(t)g(t), \quad a \in \mathfrak{g}, u \in U^\Lambda; f(t), g(t) \in \mathbb{C}[t, t^{-1}].$$

For each vector $u \in U^\Lambda$ define $u(z) = \sum_{n\in\mathbb{Z}}(u \otimes t^n)z^{-n-1}$ so that (2.7) is equivalent to

$$a(z)u(w) = au(w)\delta(z - w),$$

and hence $F_{\tilde{\mathfrak{g}}}(\Lambda)$ is conformal. This gives a family of $R(\tilde{\mathfrak{g}})$-modules, which is irreducible if and only if $\Lambda \neq 0$, in which case it will be denoted by $L_{\tilde{\mathfrak{g}}}(\Lambda)$. By $L_{\tilde{\mathfrak{g}}}(0)$ we will mean the trivial $R(\tilde{\mathfrak{g}})$-module. Similarly one defines the one-dimensional module $L_{\tilde{\mathfrak{g}}}(\alpha, 0)$.

Example 2.6. $\tilde{\mathfrak{g}}$ acts on $F_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\alpha, \Delta, \Lambda) = U^\Lambda \otimes F_{\mathfrak{V}}(\alpha, \Delta)$ similarly as in Example 2.5. However, on $F_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\alpha, \Delta, \Lambda)$ we have also an action of $\mathfrak{V}$, thus making it into a module over $\mathfrak{V} \ltimes \tilde{\mathfrak{g}}$. This module defines an $R(\mathfrak{V} \ltimes \tilde{\mathfrak{g}})$-module which is irreducible if and only if $(\Delta, \Lambda) \neq (0, 0)$, and in which case it will be denoted by $L_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\alpha, \Delta, \Lambda)$. By $L_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\alpha, 0, 0)$ we will mean the one-dimensional module on which ∂ acts a the scalar α.

The following theorem was proved in [4].

Theorem 2.1. *Let $\mathfrak{g}$ stand for a finite-dimensional simple Lie algebra. Any finite irreducible module over the conformal algebras $R(\mathfrak{V})$, $R(\tilde{\mathfrak{g}})$ and $R(\mathfrak{V} \ltimes \tilde{\mathfrak{g}})$ are as follows:*

i. $L_{\mathfrak{V}}(\alpha, \Delta)$,
ii. $L_{\tilde{\mathfrak{g}}}(\Lambda)$ *and* $L_{\tilde{\mathfrak{g}}}(\alpha, 0)$,
iii. $L_{\mathfrak{V}\ltimes\tilde{\mathfrak{g}}}(\alpha, \Delta, \Lambda)$.

Remark 2.2. We note that a similar statement as Theorem 2.1 part (iii) holds even if $\mathfrak{g}$ is replaced by the 1-dimensional Lie algebra $\mathbb{C}a$. In this case $U^\Lambda = \mathbb{C}u$ with $au = \Lambda u$, $\Lambda \in \mathbb{C}$. Also part (ii) remains true for all but three series of finite-dimensional simple Lie superalgebras.

2.3. **Extended Annihilation Subalgebras.** Given a formal distribution Lie superalgebra $(\mathfrak{g}, \mathfrak{F})$ we let $\mathfrak{g}_+$ denote the $\mathbb{C}$-span of all $a_{[n]}$, where $n \geq 0$ and $a \in \mathfrak{F}$. Due to (2.3) $\mathfrak{g}_+$ is closed under the bracket and hence form a subalgebra of $\mathfrak{g}$, which we will call the *annihilation algebra* of $(\mathfrak{g}, \mathfrak{F})$. Let ∂ be the derivation of $\mathfrak{g}_+$ defined by $[\partial, a_{[n]}] = -na_{[n-1]}$, and consider the semi-direct sum of $\mathfrak{g}^+ =$

$\mathbb{C}\partial \ltimes \mathfrak{g}_+$. Then $\mathfrak{g}^+$ is called the *extended annihilated algebra* of $(\mathfrak{g}, \mathfrak{F})$. The following proposition, which follows by comparing (M1) with (2.3), is important for the theory of conformal modules.

Proposition 2.1. [4] *Let R be a conformal superalgebra and $(\mathfrak{g}(R), R(\mathfrak{F}))$ be its associated formal distribution Lie superalgebra with extended annihilation algebra $\mathfrak{g}(R)^+$. Then a module over the conformal superalgebra R is precisely a $\mathfrak{g}(R)^+$-module M satisfying $a_{[n]}v = 0$, for each $v \in M$, $a \in R$ and $n >> 0$.*

Remark 2.3. Let R be a conformal superalgebra with $\mathbb{C}[\partial]$-basis $\{a^i | i \in I\}$ and M a free $\mathbb{C}[\partial]$-module with basis $\{v^j | j \in J\}$. Given $a^i_{(n)}v^j \in M$ for all $i \in I$, $j \in J$, $n \in \mathbb{Z}_+$, which is 0 for $n >> 0$, condition (M2) uniquely extends the action of $a^i_{(n)}$ to all of M. If in addition (M1) holds, then M is an R-module. Hence the action of an R-module M is completely determined by the action of a $\mathbb{C}[\partial]$-basis of R on a $\mathbb{C}[\partial]$-basis of M.

Example 2.7. In the case of the Virasoro algebra $\mathfrak{V}$ the annihilation algebra $\mathfrak{V}_+$ is spanned by elements L_n, $n \geq -1$. In the case of the current algebra $\tilde{\mathfrak{g}}_+$ is spanned by $a \otimes t^n$, where $a \in \mathfrak{g}$ and $n \geq 0$, while in the case of $\mathfrak{V} \ltimes \tilde{\mathfrak{g}}$ it is $\mathfrak{V}_+ \ltimes \tilde{\mathfrak{g}}_+$.

The problem of classifying conformal modules over $(\mathfrak{g}, \mathfrak{F})$ is thus reduced to the problem of classifying a class of modules over $\mathfrak{g}(\overline{\mathfrak{F}})^+$. It is clear that in all our examples one has $\mathfrak{g}(\overline{\mathfrak{F}}) = \mathfrak{g}$, and thus we are to study modules over $\mathfrak{g}^+$. Now if in addition there exists an element L_{-1} in $\mathfrak{g}_+$ such that $L_{-1} - \partial$ is central in $\mathfrak{g}^+$, then every irreducible representation of $\mathfrak{g}^+$ is an irreducible representation of $\mathfrak{g}_+$, on which $(L_{-1} - \partial)$ acts as a scalar $\alpha \in \mathbb{C}$. In the case of the $\mathfrak{V}$ and $\mathfrak{V} \ltimes \tilde{\mathfrak{g}}$ and the $N = 2, 3, 4$ superconformal superalgebras, which we will define later, such an L_{-1} always exists so that we only need to consider representations of $\mathfrak{g}_+$. The irreducible representations of $\mathfrak{V}_+$, and $\mathfrak{V}_+ \ltimes \tilde{\mathfrak{g}}_+$ that give rise to those in Theorem 2.1 are denoted by $L_{\mathfrak{V}_+}(\Delta)$ and $L_{\mathfrak{V}_+ \ltimes \tilde{\mathfrak{g}}_+}(\Delta, \Lambda)$, respectively. The corresponding actions are clear and can be found in [4].

3. Finite Verma-type Conformal Modules

Let $\mathfrak{L}$ be a Lie superalgebra over $\mathbb{C}$ with a distinguished element ∂ and a descending sequence of subspaces $\mathfrak{L} = \mathfrak{L}_{-1} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_n \supset \cdots$, such that $[\partial, \mathfrak{L}_k] = \mathfrak{L}_{k-1}$, for all $k > 0$. Let W be an $\mathfrak{L}$-module, which is finitely generated over $\mathbb{C}[\partial]$, such that for all $w \in W$ there exists a non-negative integer k (depending on w) with $\mathfrak{L}_k w = 0$. For $m \geq -2$ set $W_m = \{w \in W | \mathfrak{L}_{m+1} w = 0\}$ and let M be the minimal non-negative integer such that $W_M \neq 0$.

Lemma 3.1. [4] *Suppose that $M \geq 0$. Then $\mathbb{C}[\partial]W_M = \mathbb{C}[\partial] \otimes W_M$ and hence $\mathbb{C}[\partial]W_M \cap W_M = W_M$. In particular W_M is a finite-dimensional vector space.*

Let $\mathfrak{g}$ be a Lie superalgebra satisfying the following three conditions.

(L1) $\mathfrak{g}$ is $\mathbb{Z}$-graded of finite depth $d \in \mathbb{N}$, i.e. $\mathfrak{g} = \bigoplus_{j \geq -d} \mathfrak{g}_j$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
(L2) There exists a semisimple element $z \in \mathfrak{g}_0$ such that it centralizer in $\mathfrak{g}$ is contained in $\mathfrak{g}_0$.
(L3) There exits an element $\partial \in \mathfrak{g}_{-d}$ such that $[\partial, \mathfrak{g}_i] = \mathfrak{g}_{i-d}$, for $i \geq 0$.

Remark 3.1. If $\mathfrak{g}$ contains the grading operator with respect to its gradation, then condition (L2) is automatic.

Examples of Lie superalgebras satisfying (L1)–(L3) are provided by annihilation subalgebras of superconformal algebras, which we will describe in more detail.

Let t be an even indeterminate and $\xi_1, \ldots, \xi_N$ be N odd indeterminate. Denote by $\Lambda(N)$ the Grassmann superalgebra in the indeterminates $\xi_1, \ldots, \xi_N$ and set $\Lambda(1, N) := \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$. Let $W(1, N)$ be the derivation superalgebra of $\Lambda(1, N)$, then $W(1, N)$ is a formal distribution Lie superalgebra [8]. Letting $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \xi_i}$, for $i = 1, \ldots, N$, be the usual differential operators, every element in $D \in W(1, N)$ can be written as [10]

$$D = a_0 \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i \frac{\partial}{\partial \xi_i}, \quad a_0, a_i, \ldots, a_N \in \Lambda(1, N).$$

The *standard gradation* of $W(1, N)$ is obtained by setting the degree of t and ξ_i to be 1. Its annihilation subalgebra is $W(1, N)_+ = \bigoplus_{j \geq -1} (W(1, N))_j$. $W(1, N)_+$ in this gradation contains its grading operator given by $z = t\frac{\partial}{\partial t} + \sum_{i=1}^{N} \xi_i \frac{\partial}{\partial \xi_i}$ so that (L2) is satisfied. Also choosing ∂ to be $\frac{\partial}{\partial t}$ it follows that (L3) is also satisfied so that $W(1, N)$ is a Lie superalgebra of the type above. Note that $W(1, N)_0 \cong gl(1, N)$.

The subalgebra of divergence zero vector fields in $W(1, N)$ contains an ideal of codimension 1. This ideal is its derived algebra and is the superconformal algebra $S(1, N)$ [8]. The standard gradation of $W(1, N)_+$ induces a gradation on the annihilation subalgebra $S(1, N)_+$ of $S(1, N)$. Choosing $z = t\frac{\partial}{\partial t} + \frac{1}{N} \sum_{i=1}^{N} \xi_i \frac{\partial}{\partial \xi_i}$ along with $\partial = \frac{\partial}{\partial t}$ it follows that $S(1, N)_+$ in this gradation also satisfies (L1)–L(3). Observe that $S(1, N)_0 \cong sl(1, N)$ and also that the "small" $N = 4$ superconformal algebra (to be defined in Section 6) is isomorphic to $S(1, 2)$ [11].

The contact superalgebra $K(1, N)$ is the subalgebra of $W(1, N)$ defined by

$$K(1, N) := \{D \in W(1, N) | D\omega = f_D \omega, \text{ for some } f_D \in \Lambda(1, N)\},$$

where $\omega := dt - \sum_{i=1}^{N} \xi_i d\xi_i$ is the standard contact form. Here the action of D on ω is the usual action of vector fields on differential forms.

The map from $\Lambda(1,N)$ to $K(1,N)$ given by to

$$f \to 2f\frac{\partial}{\partial t} + (-1)^{p(f)}\sum_{i=1}^{N}(\xi_i\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi_i})(\xi_i\frac{\partial}{\partial t} + \frac{\partial}{\partial \xi_i})$$

is a bijection and hence it allows us to identify $K(1,N)$ with the polynomial superalgebra $\Lambda(1,N)$. The Lie bracket in $\Lambda(1,N)$, also called the contact bracket, then reads for homogeneous elements $f,g \in \Lambda(1,N)$:

$$[f,g] = (2-E)f\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)g + (-1)^{p(f)}\sum_{i=1}^{N}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i},$$

where $E = \sum_{i=1}^{N}\xi_i\frac{\partial}{\partial \xi_i}$ is the Euler operator.

When N is even it is sometimes more convenient to make the change of basis $\xi_j^+ = \frac{1}{\sqrt{2}}(\xi_j + i\xi_{j+\frac{N}{2}})$ and $\xi_j^- = \frac{1}{\sqrt{2}}(\xi_j - i\xi_{j+\frac{N}{2}})$, for $j = 1,\ldots,\frac{N}{2}$ and $i = \sqrt{-1}$, so that the contact bracket takes the split form:

$$[f,g] = (2-E)f\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)g + (-1)^{p(f)}\sum_{i=1}^{\frac{N}{2}}(\frac{\partial f}{\partial \xi_i^+}\frac{\partial g}{\partial \xi_i^-} + \frac{\partial f}{\partial \xi_i^-}\frac{\partial g}{\partial \xi_i^+}),$$

where E again is the Euler operator $\sum_{i=1}^{\frac{N}{2}}(\xi_i^+\frac{\partial}{\partial \xi_i^+} + \xi_i^-\frac{\partial}{\partial \xi_i^-})$.

The contact superalgebra $K(1,N)$ is a formal distribution Lie superalgebra with fields defined as follows: Let $I = \{i_1,\ldots,i_k\}$ be an ordered subset of $\{1,\ldots,N\}$, and denote by ξ_I the monomial $\xi_{i_1}\cdots\xi_{i_k}$. Each such monomial gives rise to a field $\xi_I(z) = \sum_{j\in\mathbb{Z}}\xi_I t^j z^{-j-1}$. Evidently the span of the coefficients of all such $\xi_I(z)$ is $K(1,N)$. Furthermore it is easy to check that these fields are mutually local and form a formal distribution Lie superalgebra. This Lie superalgebra becomes $\mathbb{Z}$-graded by putting the degree of $\xi_I t^n$ to $2n+k-2$. Obviously t is the grading operator of this gradation. This gradation of $K(1,N)$ is usually referred to as its *standard gradation.*

The annihilation subalgebra $K(1,N)_+$ of $K(1,N)$ is spanned by the basis elements $\xi_I t^n$, where $n \geq 0$ and I runs over all subsets of $\{i_1,\ldots,i_k\}$ ordered in (strictly) increasing order. The $\mathbb{Z}$-gradation from $K(1,N)$ induces a gradation on $K(1,N)_+$ making it a $\mathbb{Z}$-graded Lie superalgebra of depth 2 so that $K(1,N)_+ = \bigoplus_{j=-2}^{\infty}(K(1,N)_+)_j$ satisfies (L1) and (L2). In this gradation it is easy to check that $[1,(K(1,N)_+)_j] = (K(1,N)_+)_{j-2}$ for all $j \geq 0$, so that $K(1,N)_+$ also satisfies condition (L3). It is easy to see that the annihilation subalgebra of the small $N=4$ superconformal algebra, which we define in Section 6, also satisfies conditions (L1)–(L3). Note that $K(1,N)_0 \cong cso_N$, the direct sum of the Lie algebra so_N and the one-dimensional Lie algebra.

Finally it follows from the description of the exceptional superconformal algebra CK_6 as a subalgebra of $K(1,6)$ in [5] that its annihilation subalgebra $(CK_6)_+ = \bigoplus_{j\geq -2}(CK_6)_j$ is a Lie superalgebra satisfying (L1)–(L3) with $(CK_6)_0 \cong cso_6$.

The modules over the annihilation subalgebras that are equivalent to modules over the corresponding conformal superalgebras are then $\mathfrak{g}$-modules V satisfying the following conditions.

(V1) For all $v \in V$ there exists an integer $k_0 \geq -d$ (depending on v) such that $\mathfrak{g}_k v = 0$, for all $k \geq k_0$.

(V2) V is finitely generated over $\mathbb{C}[\partial]$.

We shall call $\mathfrak{g}$-modules satisfying these two properties *finite*. Let V be a finite irreducible $\mathfrak{g}$-module. For $n \geq -d-1$ set $V_n = \{v \in V | \mathfrak{g}_j v = 0, \forall j > n\}$. Let N be the minimal integer such that $V_N \neq 0$. Such an N exists by (V1).

Lemma 3.2. *If $N \geq 0$, then V_N is a finite-dimensional vector space over $\mathbb{C}$.*

Proof. We let $\mathcal{L} = \mathfrak{g}$ and put $\mathcal{L}_j = \bigoplus_{i\geq jd} \mathfrak{g}_i$ so that we have a filtration of subspaces

$$\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_n \supset \cdots,$$

with $[\partial, \mathcal{L}_i] = \mathcal{L}_{i-1}$, for all $i \geq 0$ by (L3). Let $W_m := \{v \in V | \mathcal{L}_{m+1} v = 0\}$ and let M be the minimal integer such that $W_M \neq 0$. Since $N \geq 0$ implies that $M \geq 0$, this setting puts us in the situation of Lemma 3.1, from which we conclude that W_M is a finite-dimensional vector space over $\mathbb{C}$. Of course $V_N \subset W_M$ and hence it follows that V_N is finite-dimensional as well. $\square$

We obtain the following description of finite irreducible $\mathfrak{g}$-modules.

Theorem 3.1. *Let $\mathfrak{g} = \bigoplus_{j\geq -d} \mathfrak{g}_j$ be a Lie superalgebra satisfying conditions (L1)–(L3) and V a finite irreducible $\mathfrak{g}$-module. There exists a finite-dimensional irreducible $\mathfrak{g}_0$-module U_0, extended trivially to an $\mathcal{L}_0(= \bigoplus_{j\geq 0} \mathfrak{g}_j)$-module, and a $\mathfrak{g}$-epimorphism $\varphi : \mathrm{Ind}^{\mathfrak{g}}_{\mathcal{L}_0} U_0 \to V$.*

Proof. We will continue to use the notation defined earlier. First we show that $N \leq 0$. Suppose that $N > 0$. It is easy to see that V_N is invariant under $\mathcal{L}_0$. Now there exits a basis $\{x_1, \ldots, x_m\}$ of $\mathfrak{g}_N$ together with non-zero complex number $\lambda_1, \ldots, \lambda_m$ such that $[z, x_i] = \lambda_i x$, where z is the element of (L2). Since V_N is a finite-dimensional vector space it follows in particular that x_i acts nilpotently on V_N for all $1 \leq i \leq m$. But $[\mathfrak{g}_N, \mathfrak{g}_N] \subset \bigoplus_{j\geq N+1} \mathfrak{g}_j$ and so the action of the x_i's on V_N commutes. Therefore there exits a non-zero $v \in V_N$ such that $\mathfrak{g}_N v = 0$. But in this case $V_{N-1} \neq 0$, which contradicts the minimality of N. Thus $N \leq 0$.

In the case when $N = 0$, there exists an epimorphism of $\mathfrak{g}$-modules $\mathrm{Ind}^{\mathfrak{g}}_{\mathcal{L}_0} V_0 \to V$, with V_0 finite-dimensional due to Lemma 3.2. By irreducibility of V it follows that $V_0 = U_0$ is an irreducible $\mathfrak{g}_0$-module. Now if $N < 0$, then there exists a non-zero vector v invariant under the action of $\mathfrak{g}_j$, for $j \geq 0$. Again we have an epimorphism of $\mathfrak{g}$-modules $\mathrm{Ind}^{\mathfrak{g}}_{\mathcal{L}_0} \mathbb{C}v \to V$. $\square$

As a corollary of Theorem 3.1 we obtain the following.

Corollary 3.1. *There exists a bijection between finite irreducible conformal modules of the superconformal algebra* $\mathfrak{g}$ *and finite-dimensional irreducible representations of the Lie (super)algebra* $\mathfrak{g}_0$*, where*

i. $\mathfrak{g} = K(1,N)$ *and* $\mathfrak{g}_0 = cso_N$,
ii. $\mathfrak{g} = W(1,N)$ *and* $\mathfrak{g}_0 = gl(1,N)$,
iii. $\mathfrak{g} = S(1,N)$ *and* $\mathfrak{g}_0 = sl(1,N)$,
iv. $\mathfrak{g} = CK_6$ *and* $\mathfrak{g}_0 = cso_6$.

Proof. By Theorem 3.1 every finite irreducible $\mathfrak{g}$-module is a homomorphic image of $\mathrm{Ind}^{\mathfrak{g}}_{\mathcal{L}_0}U_0$. Now the usual argument for highest weight representations implies that given a finite-dimensional irreducible $\mathfrak{g}_0$-module U_0 the $\mathfrak{g}$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathcal{L}_0}U_0$ contains a unique maximal submodule, from which the bijection then follows. $\square$

Remark 3.2. It is usual to put a half-integer gradation on $K(1,N)$ when thinking of it as a superconformal algebra. The grading operator of $K(1,N)$ with respect to this gradation is then $\frac{t}{2}$ rather than t. In this gradation one has $K(1,N)_+ = \bigoplus_{j\ge -1}\mathfrak{g}_j$, where $j \in \frac{1}{2}\mathbb{Z}$. Theorem 3.1 of course remains valid after making some obvious changes regarding gradation. For a Lie superalgebra $\mathfrak{g} = \bigoplus_{j\ge -1}\mathfrak{g}_j$ with $j \in \frac{1}{2}\mathbb{Z}$, we will make it a convention to write $\mathfrak{g}_-$ for the subalgebra $\bigoplus_{j<0}\mathfrak{g}_j$.

4. Finite irreducible Modules over the $N = 2$ conformal superalgebra

The $N = 2$ superconformal algebra is the formal distribution Lie superalgebra $K(1,2)$. Letting ξ^+, ξ^- denote the two odd indeterminates (so that we are using the split contact form) this algebra is generated by the following four fields: $L(z) = \sum_{n\in\mathbb{Z}} -\frac{t^{n+1}}{2}z^{-n-2}$, $G^{\pm}(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}}\xi^{\pm}t^{r+\frac{1}{2}}z^{-r-\frac{3}{2}}$ and $J(z) = \sum_{n\in\mathbb{Z}}\xi^-\xi^+t^nz^{-n-1}$. Its corresponding conformal superalgebra is then generated freely over $\mathbb{C}[\partial]$ by $\{L, J, G^{\pm}\}$ with products:

$$L_\lambda L = (\partial + 2\lambda)L, \quad L_\lambda J = (\partial+\lambda)J, \quad L_\lambda G^{\pm} = (\partial + \frac{3}{2}\lambda)G^{\pm},$$
$$J_\lambda G^{\pm} = \pm G^{\pm}, \quad G^+_\lambda G^- = (\partial + 2\lambda)J + 2L.$$

Letting $L_n = -\frac{t^{n+1}}{2}$, $G^{\pm}_r = \xi^{\pm}t^{r+\frac{1}{2}}$ and $J_n = \xi^-\xi^+t^n$ with $n \in \mathbb{Z}$, $r \in \frac{1}{2}+\mathbb{Z}$, the non-zero brackets in $K(1,2)$ are ($m,n \in \mathbb{Z}$ and $r,s \in \frac{1}{2}+\mathbb{Z}$):

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [L_m, G^{\pm}_r] = (\frac{m}{2} - r)G^{\pm}_{m+r}, \quad [L_m, J_n] = -nJ_{n+m},$$
$$[J_m, G^{\pm}_r] = \pm G^{\pm}_{m+r}, \qquad [G^+_r, G^-_s] = 2L_{r+s} + (r-s)J_{r+s}.$$

The annihilation subalgebra $\mathfrak{g} = K(1,2)_+$ is then spanned by L_m, J_n and $G^{\pm}_r$, where $m \ge -1$, $n \ge 0$ and $r \ge -\frac{1}{2}$. Note that letting $\mathfrak{g}_j$ be the span of X_j, where

$X = L, J, G^{\pm}$, equips $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$, $j \in \frac{1}{2}\mathbb{Z}$, with a (consistent) $\frac{1}{2}\mathbb{Z}$-gradation. We denote L_{-1} by ∂ from now on.

Let $\mathbb{C}v_{\Delta,\Lambda}$, $\Delta, \Lambda \in \mathbb{C}$, be the one-dimensional module over the abelian Lie algebra $\mathfrak{g}_0 = \mathbb{C}L_0 + \mathbb{C}J_0$, determined by

$$L_0 v_{\Delta,\Lambda} = \Delta v_{\Delta,\Lambda}, \quad J_0 v_{\Delta,\Lambda} = \Lambda v_{\Delta,\Lambda}.$$

We may extend $\mathbb{C}v_{\Delta,\Lambda}$ to a module over $\mathfrak{L}_0 = \bigoplus_{j \geq 0} \mathfrak{g}_j$ by setting $\mathfrak{g}_j v_{\Delta,\Lambda} = 0$, for $j > 0$. Let $M_{\mathfrak{N}^2_+}(\Delta, \Lambda) := \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{L}_0} \mathbb{C}v_{\Delta,\Lambda}$. We denote by N the unique maximal submodule of $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$. The quotient $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)/N$ is the irreducible highest weight module $L_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ of highest weight (Δ, Λ). By Theorem 3.1 $L_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ for $\Delta, \Lambda \in \mathbb{C}$ form a complete list of finite irreducible $K(1, 2)_+$-modules. Our next objective is to give a more explicit description of N and hence of $L_{\mathfrak{N}^2_+}(\Delta, \Lambda)$.

It is clear that $\partial^k v_{\Delta,\Lambda}$, $\partial^k G^+_{-\frac{1}{2}} v_{\Delta,\Lambda}$, $\partial^k G^-_{-\frac{1}{2}} v_{\Delta,\Lambda}$ and $\partial^k G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} v_{\Delta,\Lambda}$, $k \geq 0$, is a basis consisting of (L_0, J_0)-weight vectors for $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ of (L_0, J_0)-weights $(\Delta + k, \Lambda)$, $(\Delta + k + \frac{1}{2}, \Lambda + 1)$, $(\Delta + k + \frac{1}{2}, \Lambda - 1)$ and $(\Delta + k + 1, \Lambda)$, respectively. A non-zero (L_0, J_0)-weight vector $v \in M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ is called a *singular vector* if $\mathfrak{g}_j v = 0$, for all $j > 0$. We call a singular vector *proper* if it is not a scalar multiple of the highest weight vector $v_{\Delta,\Lambda}$. Obviously $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ is irreducible if and only if $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ contains no proper singular vector. We now analyze singular vectors inside $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$.

Lemma 4.1. *Let $k \geq 1$ and suppose that $w = \alpha\partial^k v_{\Delta,\Lambda} + \beta\partial^{k-1} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} v_{\Delta,\Lambda}$ is a singular vector of (L_0, J_0)-weight $(\Delta + k, \Lambda)$ in $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$, where $\alpha, \beta \in \mathbb{C}$. Then $k = 1$. Furthermore any proper singular vector of this form is a scalar multiple of either $G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} v_{\Delta,\Lambda}$, in which case $\Delta = -\frac{1}{2}$ and $\Lambda = 1$, or $(-2\partial + G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}}) v_{\Delta,\Lambda}$, in which case $\Delta = -\frac{1}{2}$ and $\Lambda = -1$.*

Proof. Note that w is singular if and only if $J_1 w = G^{\pm}_{\frac{1}{2}} w = 0$. We compute

$$G^+_{\frac{1}{2}} w = (\alpha k - \beta(2\Delta + \Lambda))\partial^{k-1} G^+_{-\frac{1}{2}} v_{\Delta,\Lambda} = 0, \tag{4.1}$$

$$G^-_{\frac{1}{2}} w = (\alpha k + \beta(2\Delta - \Lambda + 2k))\partial^{k-1} G^-_{-\frac{1}{2}} v_{\Delta,\Lambda} = 0, \tag{4.2}$$

$$J_1 w = (\alpha\Lambda k + \beta(2\Delta + \Lambda))\partial^{k-1} v_{\Delta,\Lambda} + \beta(k-1)\Lambda\partial^{k-2} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} v_{\Delta,\Lambda} = 0. \tag{4.3}$$

But then $\beta \neq 0$, since otherwise (4.1) would imply that $k = 0$. However, $\beta \neq 0$ together with (4.1) and (4.2) implies that

$$2\Delta + k = 0. \tag{4.4}$$

Now (4.3) gives

$$\alpha\Lambda k + \beta(2\Delta + \Lambda) = 0, \quad \beta(k-1)\Lambda = 0. \tag{4.5}$$

Now if $k>1$, then (4.5) gives $\Lambda=0$ and $\Delta=0$. But then $k=0$ by (4.1). Hence $k=1$ so that by (4.4) we have $\Delta=-\frac{1}{2}$.

Now if $\alpha\neq 0$, we have from (4.1) and (4.3) $\alpha(1+\Lambda)=0$ and hence $\Lambda=-1$. The first equation of (4.5) then implies that $\alpha+2\beta=0$.

On the other hand if $\alpha=0$, the first equation of (4.5) gives $\Lambda=1$. $\square$

Lemma 4.2. *Let $k\in\mathbb{Z}_+$.*

i. *If $\partial^k G^+_{-\frac{1}{2}}v_{\Delta,\Lambda}$ is a singular vector of (L_0,J_0)-weight $(\Delta+k+\frac{1}{2},\Lambda+1)$, then $k=0$ and $2\Delta-\Lambda=0$. Furthermore in this case $G^+_{-\frac{1}{2}}v_{\Delta,\Lambda}$ is a singular vector.*

ii. *If $\partial^k G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}$ is a singular vector of (L_0,J_0)-weight $(\Delta+k+\frac{1}{2},\Lambda-1)$, then $k=0$ and $2\Delta+\Lambda=0$. Furthermore in this case $G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}$ is a singular vector.*

Proof. The lemma follows immediately from the following two equations:

$$G^-_{\frac{1}{2}}\partial^k G^+_{-\frac{1}{2}}v_{\Delta,\Lambda}=(2\Delta-\Lambda+2k)\partial^k v_{\Delta,\Lambda}+k\partial^{k-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}=0,$$
$$G^+_{\frac{1}{2}}\partial^k G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}=(2\Delta+\Lambda)\partial^k v_{\Delta,\Lambda}+k\partial^{k-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}=0.$$

$\square$

Thus Lemma 4.1 and Lemma 4.2 prove the following.

Proposition 4.1. *Any proper singular vector in $M_{\mathfrak{N}^2_+}(\Delta,\Lambda)$ is a scalar multiple of*

i. *$G^+_{-\frac{1}{2}}v_{\Delta,\Lambda}$, in which case we have $2\Delta-\Lambda=0$. In the particular case of $\Delta=-\frac{1}{2}$ and $\Lambda=-1$ we have in addition $G^-_{-\frac{1}{2}}G^+_{-\frac{1}{2}}v_{\Delta,\Lambda}$.*

ii. *$G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}$, in which case we have $2\Delta+\Lambda=0$. In the particular case of $\Delta=-\frac{1}{2}$ and $\Lambda=1$ we have in addition $G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}$.*

Let N be the subspace of $M_{\mathfrak{N}^2_+}(\Delta,\Lambda)$ given by

$$N=\mathbb{C}[\partial]G^+_{-\frac{1}{2}}v_{\Delta,\Lambda}+\mathbb{C}[\partial]G^-_{-\frac{1}{2}}G^+_{-\frac{1}{2}}v_{\Delta,\Lambda},\quad \text{if } 2\Delta-\Lambda=0 \text{ and } \Lambda\neq 0,$$
$$N=\mathbb{C}[\partial]G^-_{-\frac{1}{2}}v_{\Delta,\Lambda}+\mathbb{C}[\partial]G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}v_{\Delta,\Lambda},\quad \text{if } 2\Delta+\Lambda=0 \text{ and } \Lambda\neq 0.$$

It follows from Proposition 4.1 that in either case N is a submodule of $M_{\mathfrak{N}^2_+}(\Delta,\Lambda)$.

Theorem 4.1. *The modules $L_{\mathfrak{N}^2_+}(\Delta,\Lambda)$, for $\Delta,\Lambda\in\mathbb{C}$, form a complete list of non-isomorphic finite (over $\mathbb{C}[\partial]$) irreducible $K(1,2)_+$-modules. Furthermore $L_{\mathfrak{N}^2_+}(\Delta,\Lambda)$ as a $\mathbb{C}[\partial]$-module has rank*

i. *4, in the case $2\Delta\pm\Lambda\neq 0$,*

ii. *2, in the case $2\Delta\pm\Lambda=0$ and $2\Delta\mp\Lambda\neq 0$,*

iii. 0, *in the case* $\Delta = \Lambda = 0$.

Proof. If $2\Delta+\Lambda \neq 0$ and $2\Delta-\Lambda \neq 0$, then by Proposition 4.1 $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ contains no proper singular vector and hence is irreducible.

Suppose that $2\Delta + \Lambda = 0$ and $2\Delta - \Lambda \neq 0$. In this case consider the submodule of $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ generated by the singular vector $G^-_{-\frac{1}{2}} v_{\Delta,\Lambda}$. This module is precisely N above and hence $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)/N$ is freely generated over $\mathbb{C}[\partial]$ by $v_{\Delta,\Lambda}$ and $G^+_{-\frac{1}{2}} v_{\Delta,\Lambda}$. We claim that $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)/N$ is irreducible. The even part of $K(1, 2)_+$ is isomorphic to the semi-direct sum of $\mathfrak{V}_+$ (generated by L_n) and $\tilde{\mathfrak{g}}_+$ (generated by J_n), where $\mathfrak{g}$ is the one-dimensional Lie algebra. We first consider $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)/N$ as a module over the $\mathfrak{V}_+ \ltimes \tilde{\mathfrak{g}}_+$. The vectors $v_{\Delta,\Lambda}$ and $G^+_{\frac{1}{2}} v_{\Delta,\Lambda}$ have (L_0, J_0)-weights (Δ, Λ) and $(\Delta + \frac{1}{2}, \Lambda + 1)$, respectively, and furthermore are both annihilated by L_n and J_n, for $n \geq 1$. Now since $2\Delta + \Lambda = 0$ and $2\Delta - \Lambda \neq 0$, we have $(\Delta, \Lambda) \neq (0, 0)$ and $(\Delta + \frac{1}{2}, \Lambda + 1) \neq (0, 0)$. From this it follows that $M_{\mathfrak{N}^2_+}(\Delta, \Lambda)/N$ as a module over $\mathfrak{V}_+ \ltimes \tilde{\mathfrak{g}}_+$ is a direct sum of two non-isomorphic irreducible modules, namely $\mathbb{C}[\partial]v_{\Delta,\Lambda} \cong L_{\mathfrak{V}_+\ltimes\tilde{\mathfrak{g}}_+}(\Delta, \Lambda)$ and $\mathbb{C}[\partial]G^+_{-\frac{1}{2}} v_{\Delta,\Lambda} \cong L_{\mathfrak{V}_+\ltimes\tilde{\mathfrak{g}}_+}(\Delta + \frac{1}{2}, \Lambda + 1)$ (see Section 2 for notation). But we have

$$G^-_{\frac{1}{2}} G^+_{-\frac{1}{2}} v_{\Delta,\Lambda} = (2\Delta - \Lambda)v_{\Delta,\Lambda} \neq 0,$$

which implies that as a $K(1, 2)_+$-module $L_{\mathfrak{N}^2_+}(\Delta, \Lambda)$ is irreducible.

The case when $2\Delta - \Lambda = 0$ and $2\Delta + \Lambda \neq 0$ is completely analogous and we leave it to the reader.

Finally in the case when $\Delta = \Lambda = 0$, both $G^+_{-\frac{1}{2}} v_{\Delta,\Lambda}$ and $G^-_{-\frac{1}{2}} v_{\Delta,\Lambda}$ are proper singular vectors. Now the submodule in $M_{\mathfrak{N}^2_+}(0, 0)$ generated by these two vectors contains $[G^+_{-\frac{1}{2}}, G^-_{-\frac{1}{2}}]v_{\Delta,\Lambda} = 2\partial v_{\Delta,\Lambda}$, and hence has codimension 1 over $\mathbb{C}$. So the resulting quotient is the trivial module. □

It follows that every finite irreducible module over the $N = 2$ conformal superalgebra is of the form $L_{\mathfrak{N}^2}(\alpha, \Delta, \Lambda)$, where $\alpha, \Delta, \Lambda \in \mathbb{C}$. We will write down explicit formulas for the action of the conformal superalgebra on such irreducible modules in the generating series form. Since we have already explained in Section 2 how such formulas can be obtained in general, we will omit the proofs.

In the case when $2\Delta \pm \Lambda \neq 0$ the module $L_{\mathfrak{N}^2}(\alpha, \Delta, \Lambda)$ is generated freely over $\mathbb{C}[\partial]$ by two even vectors v, v^{+-} and two odd vectors v^+, v^-. We have the

following action on the generators:

$$
\begin{aligned}
&L_\lambda v = (\partial + \alpha + \Delta\lambda)v, \quad L_\lambda v^{\pm} = (\partial + \alpha + (\Delta + \frac{1}{2})\lambda)v^{\pm},\\
&L_\lambda v^{+-} = (\partial + \alpha + (\Delta+1)\lambda)v^{+-} + (\Delta + \frac{\Lambda}{2})\lambda^2 v,\\
&J_\lambda v = \Lambda v, \quad J_\lambda v^{\pm} = (\Lambda \pm 1)v^{\pm}, \quad J_\lambda v^{+-} = \Lambda v^{+-} + (2\Delta + \Lambda)\lambda v,\\
&G^{\pm}_\lambda v = v^{\pm}, \quad G^{+}_\lambda v^{+} = G^{-}_\lambda v^{-} = 0, \quad G^{+}_\lambda v^{-} = v^{+-} + (2\Delta + \Lambda)\lambda v,\\
&G^{+}_\lambda v^{+-} = -\lambda(2\Delta + \Lambda)v^{+}, \quad G^{-}_\lambda v^{+} = (2\partial + 2\alpha + \lambda(2\Delta - \Lambda))v - v^{+-},\\
&G^{-}_\lambda v^{+-} = (2\partial + 2\alpha + (2\Delta + 2 - \Lambda)\lambda)v^{-}.
\end{aligned}
$$

In the case when $2\Delta + \Lambda = 0$ but $2\Delta - \Lambda \neq 0$ the module $L_{\mathfrak{N}^2}(\alpha, \Delta, \Lambda)$ is generated freely over $\mathbb{C}[\partial]$ by one even vector v and one odd vector v^{+}. The action is then given by

$$
\begin{aligned}
&L_\lambda v = (\partial + \alpha + \Delta\lambda)v, \quad L_\lambda v^{+} = (\partial + \alpha + (\Delta + \frac{1}{2})\lambda)v^{+},\\
&J_\lambda v = -2\Delta v, \quad J_\lambda v^{+} = (-2\Delta + 1)v^{+}, \quad G^{+}_\lambda v = v^{+}, \quad G^{+}_\lambda v^{+} = 0,\\
&G^{-}_\lambda v = 0, \quad G^{-}_\lambda v^{+} = (2\partial + 2\alpha + 4\Delta\lambda)v.
\end{aligned}
$$

In the case $2\Delta - \Lambda = 0$ but $2\Delta + \Lambda \neq 0$ the module $L_{\mathfrak{N}^2}(\alpha, \Delta, \Lambda)$ is generated freely over $\mathbb{C}[\partial]$ by one even vector v and one odd vector v^{-} with action:

$$
\begin{aligned}
&L_\lambda v = (\partial + \alpha + \Delta\lambda)v, \quad L_\lambda v^{-} = (\partial + \alpha + (\Delta + \frac{1}{2})\lambda)v^{-},\\
&J_\lambda v = 2\Delta v, \quad J_\lambda v^{-} = (2\Delta - 1)v^{-}, \quad G^{+}_\lambda v = 0,\\
&G^{+}_\lambda v^{-} = (2\partial + 2\alpha + 4\Delta\lambda)v, \quad G^{-}_\lambda v = v^{-}, \quad G^{-}_\lambda v^{-} = 0.
\end{aligned}
$$

Finally $L_{\mathfrak{N}^2}(\alpha, 0, 0)$ is the one-dimensional trivial module on which ∂ acts as the scalar α.

Remark 4.1. We note that the formulas above are obtained by first putting $v = v_{\Delta,\Lambda}$, $v^{\pm} = G^{\pm}_{-\frac{1}{2}} v_{\Delta,\Lambda}$ and $v^{+-} = G^{+}_{-\frac{1}{2}} G^{-}_{-\frac{1}{2}} v_{\Delta,\Lambda}$ and then compute the action of the operators L_n, J_m and $G^{\pm}_r$, for $n \geq -1$, $m \geq 0$ and $r \geq -\frac{1}{2}$ on these vector. Translation into the language of conformal modules is an easy task using these formulas and we will omit this. Of course the parity of the vectors $v, v^{\pm}, v^{+-}$ in all the examples above can be reversed. Finally we note that the adjoint module is isomorphic to $L_{\mathfrak{N}^2}(0, 1, 0)$.

5. Finite irreducible Modules over the $N = 3$ conformal superalgebra

The $N = 3$ superconformal algebra is the formal distribution Lie superalgebra $K(1, 3)$. Letting ξ_1, ξ_2, ξ_3 be the three odd indeterminates $K(1, 3)$ is spanned over

$\mathbb{C}$ by the following basis elements ($n \in \mathbb{Z}$ and $r \in \frac{1}{2} + \mathbb{Z}$):

$$L_n = -\frac{t^{n+1}}{2}, \quad H_n = 2i\xi_1\xi_2 t^n, \quad E_n = (-\xi_1\xi_3 - i\xi_2\xi_3)t^n, \quad F_n = (\xi_1\xi_3 - i\xi_2\xi_3)t^n,$$
$$\Psi_r = -\xi_1\xi_2\xi_3 t^{r-\frac{1}{2}} \quad h_r = -2i\xi_3 t^{r+\frac{1}{2}}, \quad e_r = (i\xi_1 - \xi_2)t^{r+\frac{1}{2}}, \quad f_r = (i\xi_1 + \xi_2)t^{r+\frac{1}{2}}.$$

Let $\{H, E, F\}$ denote the standard basis of the Lie algebra sl_2 and $\{h, e, f\}$ denote the standard basis of its adjoint module. Furthermore we let $(\cdot|\cdot)$ denote the non-degenerate invariant symmetric bilinear form on sl_2 with $(H|H) = 2$. Keeping this notation in mind the commutation relations of $K(1, 3)$ are then given as follows (where $X, Y = H, E, F$ and $x, y = h, e, f$):

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, X_n] = -nX_{m+n}, \quad [L_m, x_r] = (\frac{m}{2} - r)x_{m+r},$$
$$[L_m, \Psi_r] = (-\frac{m}{2} - r)\Psi_{m+r}, \quad [X_m, Y_n] = [X, Y]_{m+n}, \quad [X_m, \Psi_r] = 0,$$
$$[X_m, y_r] = [X, y]_{m+r} + 2m(X|Y)\Psi_{m+r}, \quad [x_r, \Psi_s] = -X_{r+s}, \quad [\Psi_r, \Psi_s] = 0,$$
$$[x_r, y_s] = -(r - s)[X, Y]_{r+s} - 4(X|Y)L_{r+s},$$

where $m, n \in \mathbb{Z}$ and $r, s \in \frac{1}{2} + \mathbb{Z}$. Above we have written $[X, y]$ for the action of X on y. The eight formal distributions generating this algebra are given by $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}$, $x(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} x_r z^{-r-\frac{3}{2}}$ and $\Psi(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \Psi_r z^{-r-\frac{1}{2}}$. The corresponding operator product expansions of these fields are easily derived from (2.3), and so we will omit them.

The annihilation subalgebra $K(1, 3)_+$ is equipped with a $\frac{1}{2}\mathbb{Z}$-gradation of depth 1, i.e. $K(1, 3)_+ = \mathfrak{g} = \bigoplus_{j\geq -1} \mathfrak{g}_j$, $j \in \frac{1}{2}\mathbb{Z}$, and its 0-th graded component $\mathfrak{g}_0$ is isomorphic to a copy of $gl_2 \cong sl_2 \oplus \mathbb{C}L_0$, with H_0, E_0 and F_0 providing the standard basis for the copy of sl_2.

Let $U^{\Delta,\Lambda}$ be the finite-dimensional irreducible sl_2-module of highest weight $\Lambda \in \mathbb{Z}_+$ on which L_0 acts as the scalar Δ. We let $v_{\Delta,\Lambda}$ be a highest weight vector in $U^{\Delta,\Lambda}$. We extend $U^{\Delta,\Lambda}$ to a module over the subalgebra $\mathcal{L}_0 = \bigoplus_{j\geq 0} \mathfrak{g}_j$ in a trivial way and call this $\mathcal{L}_0$-module also $U^{\Delta,\Lambda}$. By Theorem 3.1 every finite irreducible $\mathfrak{g}$-module is a homomorphic image of $M_{\mathfrak{N}^3_+}(\Delta, \Lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathcal{L}_0} U^{\Delta,\Lambda}$ and furthermore $M_{\mathfrak{N}^3_+}(\Delta, \Lambda)$ has a unique maximal submodule N, whose irreducible quotient we denote by $L_{\mathfrak{N}^3_+}(\Delta, \Lambda)$.

Note that $M_{\mathfrak{N}^3_+}(\Delta, \Lambda)$ as a module over sl_2 is a direct sum of infinitely many copies of finite-dimensional irreducible representations. Since ∂ commutes with E_0, the E_0-invariants $M_{\mathfrak{N}^3_+}(\Delta, \Lambda)^{E_0}$ is a $\mathbb{C}[\partial]$-submodule of $M_{\mathfrak{N}^3_+}(\Delta, \Lambda)$, and hence is a free $\mathbb{C}[\partial]$-module. We can write down explicitly formulas for a $\mathbb{C}[\partial]$-basis of $M_{\mathfrak{N}^3_+}(\Delta, \Lambda)^{E_0}$. In the case when $\Lambda \geq 2$ the following is a $\mathbb{C}[\partial]$-basis:

$$a_1 = v_{\Delta,\Lambda}, \quad a_2 = e_{-\frac{1}{2}} v_{\Delta,\Lambda}, \quad a_3 = (\Lambda h_{-\frac{1}{2}} + 2e_{-\frac{1}{2}} F_0)v_{\Delta,\Lambda},$$

$$a_4 = ((\Lambda-1)(\Lambda f_{-\frac12} - h_{-\frac12}F_0) - e_{-\frac12}F_0^2)v_{\Delta,\Lambda},$$
$$a_5 = e_{-\frac12}h_{-\frac12}v_{\Delta,\Lambda}, \quad a_6 = (\Lambda e_{-\frac12}f_{-\frac12} - e_{-\frac12}h_{-\frac12}F_0)v_{\Delta,\Lambda},$$
$$a_7 = ((\Lambda-1)(\Lambda h_{-\frac12}f_{-\frac12} + 4\partial F_0 + 2e_{-\frac12}f_{-\frac12}F_0) - e_{-\frac12}h_{-\frac12}F_0^2)v_{\Delta,\Lambda},$$
$$a_8 = (e_{-\frac12}h_{-\frac12}f_{-\frac12} - 2\partial h_{-\frac12})v_{\Delta,\Lambda}.$$

The cases $\Lambda = 0, 1$ are similar. Namely, when $\Lambda = 1$ we have $a_4 = a_7 = 0$, and the remaining 6 vectors form a $\mathbb{C}[\partial]$-basis. Finally, in the case when $\Lambda = 0$, the terms $a_3 = a_4 = a_6 = a_7 = 0$, so that $M_{\mathfrak{N}^3_+}(\Delta, 0)^{E_0}$ has rank 4 over $\mathbb{C}[\partial]$. (Actually the vectors a_i depend on Λ, so it would be more appropriate to write something like a_i^Λ instead of just a_i. However, from the context it will always be clear what Λ is, so that it is safe to adopt the simpler notation of a_i.) We denote the coefficient of $v_{\Delta,\Lambda}$ in the expression a_i by u_i^Λ so that we have $a_i = u_i^\Lambda v_{\Delta,\Lambda}$, for $i = 1, \ldots, 8$. For example $u_1^\Lambda = 1$, while $u_2^\Lambda = e_{-\frac12}$ etc. We note that finding all vectors in $M_{\mathfrak{N}^3_+}(\Delta,\Lambda)^{E_0}$ above amounts essentially to decomposing tensor products of irreducible representations of sl_2 and then finding the corresponding highest weight vectors of the irreducible components.

Similarly we call a non-zero (L_0, H_0)-weight vector v in $M_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ *singular* if $v \in M_{\mathfrak{N}^3_+}(\Delta,\Lambda)^{E_0}$ and $\mathfrak{g}_j v = 0$, for all $j > 0$. As before a singular vector is called *proper* if it is not a scalar multiple of $v_{\Delta,\Lambda}$. Evidently $M_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ is irreducible if and only if $M_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ contains no proper singular vector. Our first objective is to classify singular vectors inside $M_{\mathfrak{N}^3_+}(\Delta,\Lambda)$.

Proposition 5.1. *Any proper singular vector in $M_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ is of the form ($\alpha \in \mathbb{C}$ with $\alpha \neq 0$)*

i. αa_2, *if* $4\Delta - \Lambda = 0$,
ii. αa_4, *if* $4\Delta + \Lambda + 2 = 0$ *and* $\Lambda \geq 2$,
iii. αa_6, *if* $4\Delta + \Lambda + 2 = 0$ *and* $\Lambda = 1$.

Remark 5.1. The proof of the proposition is a straightforward, albeit a tedious, calculation. We will not give the details here, but instead just point out that a weight vector $v \in M_{\mathfrak{N}^3_+}(\Delta,\Lambda)^{E_0}$ is singular if and only if $f_{\frac12}$ and $\Psi_{\frac12}$ annihilates v. This fact simplifies the calculation significantly.

From Proposition 5.1 one obtains immediately the following.

Corollary 5.1. *Suppose that (Δ,Λ) does not satisfy either $4\Delta - \Lambda = 0$ or $4\Delta + \Lambda + 2 = 0$ and $\Lambda \geq 1$. Then $L_{\mathfrak{N}^3_+}(\Delta,\Lambda) = M_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ is an irreducible $K(1,3)_+$-module of rank $8\Lambda + 8$ over $\mathbb{C}[\partial]$.*

Proposition 5.2. *Suppose that $4\Delta - \Lambda = 0$. Then $L_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ is a free $\mathbb{C}[\partial]$-module of rank 4Λ.*

Proof. By Proposition 5.1 a_2 is a singular vector in $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)$ of H_0-weight $\Lambda+2$. Consider N, the $\mathfrak{g}$-submodule generated by a_2. Then we have $N = U(\mathfrak{g}_-)V_2$, where V_2 is the irreducible sl_2-submodule generated by a_2. Note that the map $v_{\frac{\Lambda}{4}+\frac{1}{2},\Lambda+2} \to a_2$ extends uniquely to an epimorphism of $K(1,3)_+$-modules from $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4} + \frac{1}{2}, \Lambda + 2)$ to N. In particular it is an sl_2-module epimorphism. Now both modules are completely reducible sl_2-modules and hence this map sends E_0-invariants onto E_0-invariants. Since $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4} + \frac{1}{2}, \Lambda + 2)^{E_0}$ is generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda+2} v_{\frac{\Lambda}{4}+\frac{1}{2},\Lambda+2} | 1 \leq i \leq 8\}$, it follows that N^{E_0} is generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda+2} a_2 | 1 \leq i \leq 8\}$. Now N^{E_0} is a $\mathbb{C}[\partial]$-submodule of $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)$, since $[\partial, E_0] = 0$. Thus it is a free $\mathbb{C}[\partial]$-submodule generated by $\{u_i^{\Lambda+2} a_2 | 1 \leq i \leq 8\}$. We compute

$$\begin{aligned}
&u_1^{\Lambda+2}a_2 = a_2, \quad u_2^{\Lambda+2}a_2 = 0, \quad u_3^{\Lambda+2}a_2 = -(\Lambda+4)a_5,\\
&u_4^{\Lambda+2}a_2 = -(\Lambda+3)a_6 - 4(\Lambda+1)(\Lambda+3)\partial a_1, \quad u_5^{\Lambda+2}a_2 = 0,\\
&u_6^{\Lambda+2}a_2 = -4(\Lambda+3)\partial a_2, \quad u_7^{\Lambda+2}a_2 = (\Lambda+3)(\Lambda+2)a_8 + 2(\Lambda+3)\partial a_3,\\
&u_8^{\Lambda+2}a_2 = -2\partial a_5.
\end{aligned}$$

By inspection it is clear that the following is a set of $\mathbb{C}[\partial]$-generators for N^{E_0}.

$$S^\Lambda = \{a_2, a_5, a_6 + 4(\Lambda+1)\partial a_1, a_8 + \frac{2}{\Lambda+2}\partial a_3\}.$$

First consider the case when $\Lambda \geq 2$. It follows from the description of S^λ above that $\{a_1, a_3, a_4, a_7\}$ is a $\mathbb{C}[\partial]$-basis for the E_0-invariants of the quotient $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N$. Since a_1 and a_3 both have H_0-weight Λ, they generate two copies of the irreducible sl_2-module of dimension $\Lambda + 1$. On the other hand a_4 and a_7 both have weight $\Lambda - 2$, and so they generate two copies of the irreducible sl_2-module of dimension $\Lambda - 1$. Thus $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N$ is a free $\mathbb{C}[\partial]$-module of rank $2(\Lambda+1) + 2(\Lambda-1) = 4\Lambda$. So in order to complete the proof it remains to show that $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N$ is irreducible.

Note that L_n, $n \geq -1$, together with E_0, H_0, F_0 generate a copy of $\mathfrak{V}_+ \oplus sl_2$ and so we may consider $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N$ as a module over $\mathfrak{V}_+ \oplus sl_2$. By parity consideration $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N$ is a direct sum of two $(\mathfrak{V}_+ \oplus sl_2)$-modules, namely $(M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N)_{\bar{0}} = \mathbb{C}[\partial]V_1 + \mathbb{C}[\partial]V_7$ and $(M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N)_{\bar{1}} = \mathbb{C}[\partial]V_3 + \mathbb{C}[\partial]V_4$, where V_i is the irreducible sl_2-module generated by a_i. It is subject to a direct verification that L_n, for $n \geq 1$, annihilates the vectors a_1, a_3, a_4, a_7 (in fact one only needs to check that $L_1 a_7 = 0$, others being trivial) and hence $M_{\mathfrak{N}^3_+}(\frac{\Lambda}{4}, \Lambda)/N$ as a $\mathfrak{V}_+ \oplus sl_2$-module is a direct sum the following four non-isomorphic irreducible modules: $\mathbb{C}[\partial]V_3 \cong L_{\mathfrak{V}_+}(\frac{\Lambda}{4} + \frac{1}{2}) \boxtimes U^\Lambda$, $\mathbb{C}[\partial]V_4 \cong L_{\mathfrak{V}_+}(\frac{\Lambda}{4} + \frac{1}{2}) \boxtimes U^{\Lambda-2}$, $\mathbb{C}[\partial]V_1 \cong L_{\mathfrak{V}_+}(\frac{\Lambda}{4}) \boxtimes U^\Lambda$ and $\mathbb{C}[\partial]V_7 \cong L_{\mathfrak{V}_+}(\frac{\Lambda}{4} + 1) \boxtimes U^{\Lambda-2}$, where we denote by U^μ the

irreducible sl_2-module of highest weight μ. Now we compute

$$\Psi_{\frac{1}{2}}a_3 = -\Lambda(\Lambda+2)a_1, \quad f_{\frac{1}{2}}a_4 = (2\Lambda+2)F_0^2 a_1$$
$$E_1 a_7 = 2\Lambda(\Lambda-1)(2\Lambda+2)a_1, \tag{5.1}$$

from which it follows that we may go from each irreducible $\mathfrak{V}_+ \oplus sl_2$-component of $M_{\mathfrak{N}_+^3}(\frac{\Lambda}{4},\Lambda)/N$ to the irreducible component containing the highest weight vectors, and hence the module $M_{\mathfrak{N}_+^3}(\frac{\Lambda}{4},\Lambda)/N$ is irreducible.

Now if $\Lambda = 1$ the vectors a_4 and a_7 are both zero. Therefore the quotient $M_{\mathfrak{N}_+^3}(\frac{\Lambda}{4},\Lambda)/N = \mathbb{C}[\partial]V_1 \oplus \mathbb{C}[\partial]V_3$. But then the first identity in (5.1) shows that $M_{\mathfrak{N}_+^3}(\frac{\Lambda}{4},\Lambda)/N$ is irreducible. The rank of $L_{\mathfrak{N}_+^3}(\frac{\Lambda}{4},\Lambda)$ is then $2(\Lambda+1) = 4\Lambda$ in the case $\Lambda = 1$.

Finally when $\Lambda = 0$, the vectors $a_3, a_4, a_6, a_7 = 0$, so that S^Λ reduces to $\{a_2, a_5, \partial a_1, a_8\}$. Therefore $M_{\mathfrak{N}_+^3}(0,0)/N = \mathbb{C}a_1$ is the trivial module and so has rank 0. $\square$

Proposition 5.3. *Suppose that* $4\Delta + \Lambda + 2 = 0$ *and* $\Lambda \geq 1$. *Then* $L_{\mathfrak{N}_+^3}(\Delta,\Lambda)$ *is a free* $\mathbb{C}[\partial]$*-module of rank* $4\Lambda + 8$.

Proof. By Proposition 5.1 a_4 is a singular vector of $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)$ of H_0-weight $\Lambda - 2$. Let N denote the $\mathfrak{g}$-submodule generated by a_4.

Consider first the case $\Lambda \geq 4$. As in the proof of Proposition 5.1 N^{E_0} is a free $\mathbb{C}[\partial]$-module generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda-2}a_4 | 1 \leq i \leq 8\}$. We compute

$$u_1^{\Lambda-2}a_4 = a_4, \quad u_2^{\Lambda-2}a_4 = (\Lambda-1)a_6, \quad u_3^{\Lambda-2}a_4 = (\Lambda-2)a_7, \quad u_4^{\Lambda-2}a_4 = 0,$$
$$u_5^{\Lambda-2}a_4 = \Lambda(\Lambda-1)a_8 + 2(\Lambda-1)\partial a_3, \quad u_6^{\Lambda-2}a_4 = 0, \quad u_7^{\Lambda-2}a_4 = 0,$$
$$u_8^{\Lambda-2}a_4 = -2\partial a_7.$$

This implies that the set $S^\Lambda = \{a_4, (\Lambda-1)a_6, (\Lambda-2)a_7, \Lambda(\Lambda-1)a_8 + 2(\Lambda-1)\partial a_3, \partial a_7\}$ generates N^{E_0} over $\mathbb{C}[\partial]$ and so $\{a_1, a_2, a_3, a_5\}$ is a $\mathbb{C}[\partial]$-basis for $(M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N)^{E_0}$ in the case when $\Lambda \geq 4$.

Next consider the case $\Lambda = 3$. In this case, letting N be as before, N^{E_0} is generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda-2}a_4 | 1 \leq i \leq 8, i \neq 4, 7\}$. Hence it follows from the above formulas that again $\{a_1, a_2, a_3, a_5\}$ is a $\mathbb{C}[\partial]$-basis for $(M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N)^{E_0}$.

In the case when $\Lambda = 2$ we let N' denote the module generated by a_4. It follows that the vectors $\{u_1^{\Lambda-2}a_4, u_2^{\Lambda-2}a_4, u_5^{\Lambda-2}a_4, u_8^{\Lambda-2}a_4\}$ generate ${N'}^{E_0}$ over $\mathbb{C}[\partial]$ so that $S^\Lambda = \{a_4, a_6, a_8 + \partial a_3, \partial a_7\}$ generate N^{E_0}. Hence $(M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N')^{E_0}$ contains in addition a one-dimensional (over $\mathbb{C}$) subspace spanned by a_7. However, $\partial a_7 = 0$ in $(M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N')$ and hence it is a $\mathfrak{g}$-invariant by Remark 2.1. In this case we set $N = N' + \mathbb{C}a_7$ so that the quotient module $(M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N)^{E_0}$ is again generated over $\mathbb{C}[\partial]$ by $\{a_1, a_2, a_3, a_5\}$.

Now a_1 and a_3 have H_0-weight Λ, while a_2 and a_5 have H_0-weight $\Lambda+2$. Thus $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ has rank $2(\Lambda+1)+2(\Lambda+3) = 4\Lambda+8$ over $\mathbb{C}[\partial]$. So it remains to show that $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ is irreducible.

Again we consider $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ as a module over $\mathfrak{V}_+ \oplus sl_2$. It is easy to check that L_n, $n \geq 1$, annihilates a_1, a_2, a_3, a_5. (Again one really only needs to check that $L_1 a_5 = 0$.) Thus it follows that in the case of $\Lambda \geq 3$ that $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ is a direct sum of the following four non-isomorphic irreducible $\mathfrak{V}_+ \oplus sl_2$-modules: $\mathbb{C}[\partial]V_1 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+2}{4}) \boxtimes U^\Lambda$, $\mathbb{C}[\partial]V_2 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda}{4}) \boxtimes U^{\Lambda+2}$, $\mathbb{C}[\partial]V_3 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda}{4}) \boxtimes U^\Lambda$ and $\mathbb{C}[\partial]V_5 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda-2}{4}) \boxtimes U^{\Lambda+2}$, where as before U^μ stands for the irreducible sl_2-module of highest weight μ and V_i is the sl_2-submodule generated by the vector a_i. Now we compute

$$f_{\frac{1}{2}} a_2 = 2(\Lambda+1)a_1, \quad \Psi_{\frac{1}{2}} a_3 = -\Lambda(\Lambda+2)a_1, \quad F_1 a_5 = -4(\Lambda+1)a_1, \tag{5.2}$$

from which again it follows that we may go from any irreducible $\mathfrak{V}_+ \oplus sl_2$-component of $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ to the component containing the highest weight vectors, and hence $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ is irreducible.

As for the case $\Lambda = 2$ we have $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ as a $\mathfrak{V}_+ \oplus sl_2$-module is also a direct sum of the $\mathbb{C}[\partial]V_1 \oplus \mathbb{C}[\partial]V_2 \oplus \mathbb{C}[\partial]V_3 \oplus \mathbb{C}[\partial]V_5$. The first three modules, as in the case of $\Lambda \geq 3$ are irreducible. However, $\mathbb{C}[\partial]V_5$ contains a unique irreducible submodule generated by the vector ∂a_5, which is isomorphic to $L_{\mathfrak{V}_+}(-1) \boxtimes U^{\Lambda+2}$. But then (5.2) together with the fact that

$$F_2 \partial a_1 = -24 a_1$$

shows that $M_{\mathfrak{N}_+^3}(-\frac{\Lambda+2}{4},\Lambda)/N$ is irreducible in this case as well.

In the case when $\Lambda = 1$ we have by Proposition 5.1 that a_6 is the unique (up to a scalar) singular vector inside $M_{\mathfrak{N}_+^3}(-\frac{3}{4},1)$. Let N denote the $\mathfrak{g}$-submodule generated by a_6. Since a_6 has H_0-weight 1, N^{E_0} is the free $\mathbb{C}[\partial]$-module generated by $\{u_i^\Lambda a_6 | 1 \leq i \leq 8, i \neq 4, 7\}$. We have

$$u_1^\Lambda a_6 = a_6, \quad u_2^\Lambda a_6 = 0, \quad u_3^\Lambda a_6 = -3a_8 - 6\partial a_3,$$
$$u_5^\Lambda a_6 = 0, \quad u_6^\Lambda a_6 = -8\partial a_6, \quad u_8^\Lambda a_6 = -2\partial a_8 - 4\partial^2 a_3,$$

from which it follows that N^{E_0} is generated over $\mathbb{C}[\partial]$ by $S^\Lambda = \{a_6, a_8 + 2\partial a_3\}$. Since $a_4 = a_7 = 0$ in this situation, we see that $(M_{\mathfrak{N}_+^3}(-\frac{3}{4},1)/N)^{E_0}$ is generated over $\mathbb{C}[\partial]$ by the vectors a_1, a_2, a_3, a_5, just as in the case $\Lambda \geq 2$. Now the exact same argument as in the $\Lambda \geq 2$ case shows that $M_{\mathfrak{N}_+^3}(-\frac{3}{4},1)/N$ is irreducible and has rank $4\Lambda+8$ over $\mathbb{C}[\partial]$. □

We summarize the work in this section in the following theorem.

Theorem 5.1. *The modules $L_{\mathfrak{N}^3_+}(\Delta,\Lambda)$, for $\Delta\in\mathbb{C}$ and $\Lambda\in\mathbb{Z}_+$, form a complete list of non-isomorphic finite (over $\mathbb{C}[\partial]$) irreducible $K(1,3)_+$-modules. Furthermore $L_{\mathfrak{N}^3_+}(\Delta,\Lambda)$ as a $\mathbb{C}[\partial]$-module has rank*

i. 4Λ, *in the case* $4\Delta-\Lambda=0$,
ii. $4\Lambda+8$, *the case* $4\Delta+\Lambda+2=0$ *and* $\Lambda\geq 1$.
iii. $8\Lambda+8$, *in all other cases.*

Furthermore the $\mathbb{C}[\partial]$-rank of $L_{\mathfrak{N}^3_+}(\Delta,\Lambda)_{\bar{0}}$ equals the $\mathbb{C}[\partial]$-rank of $L_{\mathfrak{N}^3_+}(\Delta,\Lambda)_{\bar{1}}$ in all cases.

Remark 5.2. Translating the above theorem back into the languages of modules over conformal superalgebras and of conformal modules is now a straightforward task. We thus have proved that all finite irreducible modules over the $N=3$ conformal superalgebra are of the form $L_{\mathfrak{N}^3}(\alpha,\Delta,\Lambda)$, where $\alpha,\Delta\in\mathbb{C}$ and $\Lambda\in\mathbb{Z}_+$. The definition of these modules and also the action of the $N=3$ conformal superalgebra on them are quite easy to obtain from our explicit description of a $\mathbb{C}[\partial]$-basis of these modules. To do so would however take up quite a significant portion of space, and thus we leave this task to the interested reader. We only remark that the adjoint module is isomorphic to $L_{\mathfrak{N}^3}(0,\frac{1}{2},0)$.

6. Finite irreducible Modules over the “small” $N=4$ conformal superalgebra

The “small” $N=4$ superconformal algebra is the following subalgebra of $K(1,4)$: Let ξ_1,ξ_2,ξ_3,ξ_4 denote four odd indeterminates generating the Grassmann superalgebra $\Lambda(4)$. For a monomial ξ_I in $\Lambda(4)$ we let ξ_I^* be its Hodge dual, i.e. the unique monomial in $\Lambda(4)$ such that $\xi_I\xi_I^*=\xi_1\xi_2\xi_3\xi_4$. Then the small $N=4$ superconformal algebra is isomorphic to any of the following two subalgebras in $K(1,4)$ spanned by the following basis elements ($n\in\mathbb{Z}$, $r\in\frac{1}{2}+\mathbb{Z}$, $\beta^2=1$) [5]:

$$
\begin{aligned}
L^\beta_n&=-\frac{1}{2}(t^{n+1}+\beta n(n+1)\xi_1\xi_2\xi_3\xi_4t^{n-1}),\\
H^\beta_n&=i(\xi_1\xi_2-\beta\xi_3\xi_4)t^n,\\
E^\beta_n&=\frac{1}{2}(-\xi_1\xi_3-\beta\xi_2\xi_4-i\xi_2\xi_3+i\beta\xi_1\xi_4)t^n,\\
F^\beta_n&=\frac{1}{2}(\xi_1\xi_3+\beta\xi_2\xi_4-i\xi_2\xi_3+i\beta\xi_1\xi_4)t^n,\\
G^{-+\beta}_r&=\frac{1}{\sqrt{2}}((\xi_3+i\xi_4)t^{r+\frac{1}{2}}-\beta(r+\frac{1}{2})(\xi_3^*+i\xi_4^*)t^{r-\frac{1}{2}}),\\
G^{++\beta}_r&=\frac{1}{\sqrt{2}}((\xi_1+i\xi_2)t^{r+\frac{1}{2}}-\beta(r+\frac{1}{2})(\xi_1^*+i\xi_2^*)t^{r-\frac{1}{2}}),\\
G^{+-\beta}_r&=\frac{1}{\sqrt{2}}((\xi_3-i\xi_4)t^{r+\frac{1}{2}}-\beta(r+\frac{1}{2})(\xi_3^*-i\xi_4^*)t^{r-\frac{1}{2}}),
\end{aligned}
$$

$$G_r^{--\beta} = \frac{1}{\sqrt{2}}((i\xi_2 - \xi_1)t^{r+\frac{1}{2}} - \beta(r + \frac{1}{2})(i\xi_2^* - \xi_1^*)t^{r-\frac{1}{2}}).$$

As before let $\{H, E, F\}$ denote the standard basis of the Lie algebra sl_2 and $\{G^{++}, G^{-+}\}$ denote the standard basis of its standard module, i.e. $H \cdot G^{++} = G^{++}$, $H \cdot G^{-+} = -G^{-+}$, $E \cdot G^{++} = F \cdot G^{-+} = 0$, $F \cdot G^{++} = G^{-+}$ and $E \cdot G^{-+} = G^{++}$. Likewise $\{G^{+-}, G^{--}\}$ also denotes a copy of the standard basis of the standard sl_2-module with actions $H \cdot G^{+-} = G^{+-}$, $H \cdot G^{--} = -G^{--}$, $E \cdot G^{+-} = F \cdot G^{--} = 0$, $F \cdot G^{+-} = G^{--}$ and $E \cdot G^{--} = G^{+-}$. With this notation in mind the commutation relations are then given as follows (where $X, Y = H, E, F$ and $x, y = G^{++}, G^{-+}, G^{+-}, G^{--}$):

$$[L_m^\beta, L_n^\beta] = (m-n)L_{m+n}^\beta, \quad [L_m^\beta, X_n^\beta] = -nX_{m+n}^\beta, \quad [L_m^\beta, x_r^\beta] = (\frac{m}{2} - r)x_{m+r}^\beta,$$
$$[X_m^\beta, Y_n^\beta] = [X, Y]_{m+n}^\beta, \quad [X_m^\beta, y_r^\beta] = (X \cdot y)_{m+r}^\beta, \quad [x_r^\beta, x_s^\beta] = 0,$$
$$[G_r^{++\beta}, G_s^{+-\beta}] = (r-s)(1+\beta)E_{r+s}^\beta, \quad [G_r^{++\beta}, G_s^{-+\beta}] = (r-s)(1-\beta)E_{r+s}^\beta,$$
$$[G_r^{++\beta}, G_s^{--\beta}] = -(r-s)H_{r+s}^\beta - 2L_{r+s}^\beta, \; [G_r^{+-\beta}, G_s^{-+\beta}] = (r-s)\beta H_{r+s}^\beta + 2L_{r+s}^\beta,$$
$$[G_r^{+-\beta}, G_s^{--\beta}] = -(r-s)(1-\beta)F_{r+s}^\beta, \quad [G_r^{-+\beta}, G_s^{--\beta}] = -(r-s)(1+\beta)F_{r+s}^\beta,$$

where $m, n \in \mathbb{Z}$ and $r, s \in \frac{1}{2} + \mathbb{Z}$. The eight formal distributions generating this algebra are given by $L^\beta(z) = \sum_{n\in\mathbb{Z}} L_n^\beta z^{-n-2}$, $X^\beta(z) = \sum_{n\in\mathbb{Z}} X_n^\beta z^{-n-1}$, $x^\beta(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} x_r^\beta z^{-r-\frac{3}{2}}$. The operator product expansions of these fields are easily derived using (2.3).

We will denote the "small" $N = 4$ superconformal algebra simply by $SK(1,4)$ and assume for the rest of this section that we have chosen its realization as the subalgebra of $K(1,4)$ with $\beta = 1$ for future computational purposes. For simplicity we will drop the superscript β and write L_n for L_n^β etc. when we mean $\beta = 1$.

The annihilation subalgebra $\mathfrak{g} = SK(1,4)_+$ of $SK(1,4)$ is equipped with a $\frac{1}{2}\mathbb{Z}$-gradation of depth 1, i.e. $\mathfrak{g} = \bigoplus_{j\geq -1} \mathfrak{g}_j$, $j \in \frac{1}{2}\mathbb{Z}$, and its 0-th graded component $\mathfrak{g}_0$ is isomorphic to a copy of $gl_2 \cong sl_2 \oplus \mathbb{C}L_0$, with H_0, E_0 and F_0 providing the standard basis of the copy of sl_2. Again we let $U^{\Delta,\Lambda}$ be the finite-dimensional irreducible sl_2-module of highest weight $\Lambda \in \mathbb{Z}_+$ on which L_0 acts as the scalar Δ and $v_{\Delta,\Lambda}$ be a highest weight vector in $U^{\Delta,\Lambda}$. As in the case of $K(1,3)_+$, we may extend $U^{\Delta,\Lambda}$ to a module over the subalgebra $\mathfrak{L}_0 = \bigoplus_{j\geq 0} \mathfrak{g}_j$ trivially and call this $\mathfrak{L}_0$-module also $U^{\Delta,\Lambda}$. Again Theorem 3.1 tells us that every finite irreducible $\mathfrak{g}$-module is the quotient of $M_{\mathfrak{N}_+^4}(\Delta, \Lambda) = \mathrm{Ind}_{\mathfrak{L}_0}^{\mathfrak{g}} U^{\Delta,\Lambda}$ by its unique maximal submodule, for some $\Delta \in \mathbb{C}$ and $\Lambda \in \mathbb{Z}_+$. We denote the unique irreducible quotient by $L_{\mathfrak{N}_+^4}(\Delta, \Lambda)$ so that every finite irreducible $SK(1,4)_+$-module is of the form $L_{\mathfrak{N}_+^4}(\Delta, \Lambda)$, for $\Delta \in \mathbb{C}$ and $\Lambda \in \mathbb{Z}_+$.

Now $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)$ is completely reducible as a module over $sl_2=\mathbb{C}H_0+\mathbb{C}E_0+\mathbb{C}F_0$, and the subspace of E_0-invariants $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)^{E_0}$ is a free $\mathbb{C}[\partial]$-submodule of $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)$ due to $[E_0,\partial]=0$. We write down explicit formulas for a $\mathbb{C}[\partial]$-basis of $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)^{E_0}$, which in the case when $\Lambda\ge 2$ takes the following form:

$$
\begin{aligned}
&a_1=v_{\Delta,\Lambda},\quad a_2=G^{++}_{-\frac12}v_{\Delta,\Lambda},\quad a_3=G^{+-}_{-\frac12}v_{\Delta,\Lambda},\quad a_4=(\Lambda G^{-+}_{-\frac12}-G^{++}_{-\frac12}F_0)v_{\Delta,\Lambda},\\
&a_5=(-\Lambda G^{--}_{-\frac12}+G^{+-}_{-\frac12}F_0)v_{\Delta,\Lambda},\quad a_6=G^{-+}_{-\frac12}G^{++}_{-\frac12}v_{\Delta,\Lambda},\quad a_7=G^{+-}_{-\frac12}G^{--}_{-\frac12}v_{\Delta,\Lambda},\\
&a_8=G^{++}_{-\frac12}G^{+-}_{-\frac12}v_{\Delta,\Lambda},\quad a_9=(G^{-+}_{-\frac12}G^{+-}_{-\frac12}-G^{++}_{-\frac12}G^{--}_{-\frac12})v_{\Delta,\Lambda},\\
&a_{10}=(-\Lambda G^{++}_{-\frac12}G^{--}_{-\frac12}+G^{++}_{-\frac12}G^{+-}_{-\frac12}F_0)v_{\Delta,\Lambda},\\
&a_{11}=((\Lambda-1)(-\Lambda G^{-+}_{-\frac12}G^{--}_{-\frac12}+G^{-+}_{-\frac12}G^{+-}_{-\frac12}F_0+G^{++}_{-\frac12}G^{--}_{-\frac12}F_0)-G^{++}_{-\frac12}G^{+-}_{-\frac12}F_0^2)v_{\Delta,\Lambda},\\
&a_{12}=G^{-+}_{-\frac12}G^{++}_{-\frac12}G^{+-}_{-\frac12}v_{\Delta,\Lambda},\quad a_{13}=G^{++}_{-\frac12}G^{+-}_{-\frac12}G^{--}_{-\frac12}v_{\Delta,\Lambda},\\
&a_{14}=G^{-+}_{-\frac12}G^{++}_{-\frac12}(-\Lambda G^{--}_{-\frac12}+G^{+-}_{-\frac12}F_0)v_{\Delta,\Lambda},\\
&a_{15}=(-\Lambda G^{-+}_{-\frac12}G^{+-}_{-\frac12}G^{--}_{-\frac12}+G^{++}_{-\frac12}G^{+-}_{-\frac12}G^{--}_{-\frac12}F_0)v_{\Delta,\Lambda},\\
&a_{16}=G^{-+}_{-\frac12}G^{++}_{-\frac12}G^{+-}_{-\frac12}G^{--}_{-\frac12}v_{\Delta,\Lambda}.
\end{aligned}
$$

Now in the case when $\Lambda=1$ we have $a_{11}=0$ so that the remaining 15 vectors form a $\mathbb{C}[\partial]$-basis for $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)^{E_0}$, while in the case when $\Lambda=0$ we have $a_4=a_5=a_{10}=a_{11}=a_{14}=a_{15}=0$, so that $M_{\mathfrak{N}^4_+}(\Delta,0)^{E_0}$ has rank 10 over $\mathbb{C}[\partial]$. As in Section 5 we denote the coefficient of $v_{\Delta,\Lambda}$ in the expression a_i by u_i^Λ so that we have $a_i=u_i^\Lambda v_{\Delta,\Lambda}$, for $i=1,\dots,16$.

Singular vectors are then defined to be non-zero (L_0,H_0)-weight vectors $v\in M_{\mathfrak{N}^4_+}(\Delta,\Lambda)^{E_0}$ with $\mathfrak{g}_jv=0$, for all $j>0$. Similarly we define *proper singular vectors*. Our approach is analogous to the one of Section 5, that is first to analyze singular vectors inside $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)$. This is given by the following proposition, whose proof is again a straightforward calculation, which admittedly is rather tedious.

Proposition 6.1. *A complete list of proper singular vectors inside $M_{\mathfrak{N}^4_+}(\Delta,\Lambda)$ is given by:*

i. $2\Delta-\Lambda=0$.
  a. $\alpha a_2+\beta a_3$, $(\alpha,\beta)\neq(0,0)$,
  b. αa_8, $\alpha\neq 0$.

ii. $2\Delta+\Lambda+2=0$ *and* $\Lambda\ge 2$.
  a. $\alpha a_4+\beta a_5$, $(\alpha,\beta)\neq(0,0)$,
  b. αa_{11}, $\alpha\neq 0$.

iii. $2\Delta+\Lambda+2=0$ *and* $\Lambda=1$.
  a. $\alpha a_4+\beta a_5$, $(\alpha,\beta)\neq(0,0)$,

b. $\alpha a_{14} + \beta(a_{15} - 2\partial a_5)$, $(\alpha, \beta) \neq (0, 0)$,
c. $\alpha(a_{16} - 2\partial a_{10})$, $\alpha \neq 0$.

iv. $2\Delta + \Lambda + 2 = 0$ *and* $\Lambda = 0$.
a. $\alpha a_6 + \beta a_7 + \gamma(a_9 - 2\partial a_1)$, $(\alpha, \beta, \gamma) \neq (0, 0, 0)$,
b. $\alpha a_{13} + \beta(a_{12} + 2\partial a_2)$, $(\alpha, \beta) \neq (0, 0)$.

Remark 6.1. We note that in order to check that a weight vector $v \in M_{\mathfrak{N}_+^4}(\Delta, \Lambda)^{E_0}$ is singular, it is enough to check that v is annihilated by F_1, $G_{\frac{1}{2}}^{-+}$ and $G_{\frac{1}{2}}^{--}$.

Corollary 6.1. *Suppose that* (Δ, Λ) *does not satisfy either* $2\Delta - \Lambda = 0$ *or* $2\Delta + \Lambda + 2 = 0$. *Then* $L_{\mathfrak{N}_+^4}(\Delta, \Lambda) = M_{\mathfrak{N}_+^4}(\Delta, \Lambda)$ *is an irreducible* $SK(1, 4)_+$*-module of rank* $16\Lambda + 16$ *over* $\mathbb{C}[\partial]$.

Proposition 6.2. *Suppose that* $2\Delta - \Lambda = 0$. *Then* $L_{\mathfrak{N}_+^4}(\Delta, \Lambda)$ *is a free* $\mathbb{C}[\partial]$*-module of rank* 4Λ.

Proof. By Proposition 6.1 a_2 and a_3 are singular vectors in $M_{\mathfrak{N}_+^4}(\frac{\Lambda}{2}, \Lambda)$. Consider N_2 and N_3, the $\mathfrak{g}$-submodules generated by a_2 and a_3, respectively, and let $N = N_2 + N_3$. Then we have $N_2 = U(\mathfrak{g}_-)V_2$ and $N_3 = U(\mathfrak{g}_-)V_3$, where V_2 and V_3 are the irreducible sl_2-submodules generated by a_2 and a_3, respectively. Let's first compute $N_2^{E_0}$. Since the H_0-weight of a_2 is $\Lambda + 1$, we know that $N_2^{E_0}$ is a free $\mathbb{C}[\partial]$-module generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda+1} a_2 | 1 \leq i \leq 16\}$. We have

$$
\begin{aligned}
&u_1^{\Lambda+1} a_2 = a_2, \quad u_2^{\Lambda+1} a_2 = 0, \quad u_3^{\Lambda+1} a_2 = -a_8, \quad u_4^{\Lambda+1} a_2 = (\Lambda + 2) a_6, \\
&u_5^{\Lambda+1} a_2 = -a_9 - a_{10} + 2(\Lambda + 2)\partial a_1, \quad u_6^{\Lambda+1} a_2 = 0, \quad u_7^{\Lambda+1} a_2 = a_{13} - 2\partial a_3, \\
&u_8^{\Lambda+1} a_2 = 0, \quad u_9^{\Lambda+1} a_2 = -a_{12} + 2\partial a_2, \quad u_{10}^{\Lambda+1} a_2 = a_{12} + 2(\Lambda + 2)\partial a_2, \\
&u_{11}^{\Lambda+1} a_2 = 2(\Lambda + 2)\partial a_4 - (\Lambda + 2) a_{14}, \quad u_{12}^{\Lambda+1} a_2 = 0, \quad u_{13}^{\Lambda+1} a_2 = -2\partial a_8, \\
&u_{14}^{\Lambda+1} a_2 = 2(\Lambda + 2)\partial a_6, \quad u_{15}^{\Lambda+1} a_2 = -(\Lambda + 2) a_{16} + 2(\Lambda + 1)\partial a_9 - 2\partial a_{10}, \\
&u_{16}^{\Lambda+1} a_2 = -2\partial a_{12}.
\end{aligned}
$$

Next we find $\mathbb{C}[\partial]$-generators of $N_3^{E_0}$. Similarly $\{u_i^{\Lambda+1} a_3 | 1 \leq i \leq 16\}$ generates $N_3^{E_0}$ over $\mathbb{C}[\partial]$:

$$
\begin{aligned}
&u_1^{\Lambda+1} a_3 = a_3, \quad u_2^{\Lambda+1} a_3 = a_8, \quad u_3^{\Lambda+1} a_3 = 0, \quad u_4^{\Lambda+1} a_3 = (\Lambda + 1) a_9 - a_{10}, \\
&u_5^{\Lambda+1} a_3 = (\Lambda + 2) a_7, \quad u_6^{\Lambda+1} a_3 = a_{12}, \quad u_7^{\Lambda+1} a_3 = 0, \quad u_8^{\Lambda+1} a_3 = 0, \\
&u_9^{\Lambda+1} a_3 = a_{13}, \quad u_{10}^{\Lambda+1} a_3 = (\Lambda + 2) a_{13}, \quad u_{11}^{\Lambda+1} a_3 = -(\Lambda + 2) a_{15}, \\
&u_{12}^{\Lambda+1} a_3 = -a_{16}, \quad u_{13}^{\Lambda+1} a_3 = 0, \quad u_{14}^{\Lambda+1} a_3 = (\Lambda + 2) a_{16}, \quad u_{15}^{\Lambda+1} a_3 = 0, \\
&u_{16}^{\Lambda+1} a_3 = 0.
\end{aligned}
$$

It follows that N^{E_0} is generated over $\mathbb{C}[\partial]$ by the following set

$$S^\Lambda = \{a_2, a_3, a_6, a_7, a_8, a_9 - 2\partial a_1, a_{10} - 2(\Lambda+1)\partial a_1, a_{12}, a_{13}, a_{14} - 2\partial a_4, a_{15}, a_{16}\}.$$

Suppose that $\Lambda \geq 2$. From the description of S^Λ we see that $\{a_1, a_4, a_5, a_{11}\}$ is a $\mathbb{C}[\partial]$-basis for the E_0-invariants of the quotient $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$. Since a_1 has H_0-weight Λ, it generates a copy of the irreducible sl_2-module of dimension $\Lambda+1$. Now a_4 and a_5 both have weight $\Lambda - 1$, and so they generate two copies of the irreducible sl_2-module of dimension Λ. Finally a_{11} has weight $\Lambda - 2$, and so it generates a copy of the irreducible sl_2-module of dimension $\Lambda - 1$. Thus $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$ is a free $\mathbb{C}[\partial]$-module of rank $(\Lambda+1) + 2\Lambda + (\Lambda-1) = 4\Lambda$. So we need to show that $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$ is irreducible.

As in Section 5 L_n, $n \geq -1$, together with E_0, H_0, F_0 generate a copy of $(\mathfrak{V}_+ \oplus sl_2)$, which thus allow us to study the $(\mathfrak{V}_+ \oplus sl_2)$-module structure of $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$. By parity consideration $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$ is a direct sum of two modules, namely $(M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N)_{\bar 0} = \mathbb{C}[\partial]V_1 + \mathbb{C}[\partial]V_{11}$ and $(M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N)_{\bar 1} = \mathbb{C}[\partial]V_4 + \mathbb{C}[\partial]V_5$, where V_i is the irreducible sl_2-module generated by a_i. We can easily check that L_n, for $n \geq 1$, annihilates the vectors a_1, a_4, a_5, a_{11}. (Again the only non-trivial part is to check that $L_1 a_{11} = 0$.) Thus $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$ as a $\mathfrak{V}_+ \oplus sl_2$-module is a direct sum of the following four irreducible modules: $\mathbb{C}[\partial]V_1 \cong L_{\mathfrak{V}_+}(\frac{\Lambda}{2}) \boxtimes U^\Lambda$, $\mathbb{C}[\partial]V_4 \cong L_{\mathfrak{V}_+}(\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda-1}$, $\mathbb{C}[\partial]V_5 \cong L_{\mathfrak{V}_+}(\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda-1}$ and $\mathbb{C}[\partial]V_{11} \cong L_{\mathfrak{V}_+}(\frac{\Lambda+2}{2}) \boxtimes U^{\Lambda-2}$, where as usual U^μ is the irreducible sl_2-module of highest weight μ. Note that, contrary to the $K(1,3)_+$ case, the odd part here is a sum of two isomorphic modules. To conclude that $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{2}, \Lambda)/N$ is irreducible, we show again that one may go from each irreducible $\mathfrak{V}_+ \oplus sl_2$-component to the irreducible component containing the $\mathfrak{g}$-highest weight vectors. But this follows from the following computation.

$$G^{++}_{\frac{1}{2}}(\alpha a_4 + \beta a_5) = \beta\Lambda(2\Lambda+2)a_1, \quad \alpha, \beta \in \mathbb{C}, \tag{6.1}$$

$$G^{--}_{\frac{1}{2}} a_4 = (2\Lambda+2)F_0 a_1, \quad E_1 a_{11} = \Lambda(\Lambda-1)(2\Lambda+2)a_1. \tag{6.2}$$

Now if $\Lambda = 1$ the vector a_{11} is zero. Therefore the quotient $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{4}, \Lambda)/N = \mathbb{C}[\partial]V_1 \oplus \mathbb{C}[\partial]V_4 \oplus \mathbb{C}[\partial]V_5$. But then (6.1) and the first identity in (6.2) show that $M_{\mathfrak{N}^4_+}(\frac{\Lambda}{4}, \Lambda)/N$ is irreducible. The rank of $L_{\mathfrak{N}^4_+}(\frac{\Lambda}{4}, \Lambda)$ is then $(\Lambda+1) + 2\Lambda$, which equals to 4Λ, in the case $\Lambda = 1$.

Finally when $\Lambda = 0$, the vectors $a_4 = a_5 = a_{10} = a_{11} = a_{14} = a_{15} = 0$ so that S^Λ reduces to $\{a_2, a_3, a_6, a_7, a_8, a_9, \partial a_1, a_{12}, a_{13}, a_{16}\}$. Hence $M_{\mathfrak{N}^4_+}(0,0)/N = \mathbb{C}a_1$ is the trivial module and so has rank 0. □

Proposition 6.3. *Suppose that* $2\Delta + \Lambda + 2 = 0$. *Then* $L_{\mathfrak{N}^4_+}(\Delta, \Lambda)$ *is a free* $\mathbb{C}[\partial]$*-module of rank* $4\Lambda + 8$.

Proof. By Proposition 6.1 a_4 and a_5 are singular vectors of $M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)$ in the case $\Lambda \geq 1$.

Assume first that $\Lambda \geq 3$. Let N_4 and N_5 be the $\mathfrak{g}$-submodules generated by a_4 and a_5, respectively. We form the $\mathfrak{g}$-submodule $N = N_4 + N_5$ and consider N^{E_0}. The set $\{u_i^{\Lambda-1}a_4 | 1 \leq i \leq 16\}$ is a set of $\mathbb{C}[\partial]$-generators for $N_4^{E_0}$, since a_4 has H_0-weight $\Lambda - 1$. We have

$$\begin{aligned}
&u_1^{\Lambda-1}a_4 = a_4, \quad u_2^{\Lambda-1}a_4 = -\Lambda a_6, \quad u_3^{\Lambda-1}a_4 = 2\Lambda\partial a_1 - \Lambda a_9 + a_{10}, \quad u_4^{\Lambda-1}a_4 = 0,\\
&u_5^{\Lambda-1}a_4 = -a_{11}, \quad u_6^{\Lambda-1}a_4 = 0, \quad u_7^{\Lambda-1}a_4 = -a_{15} + 2\partial a_5, \quad u_8^{\Lambda-1}a_4 = 2\Lambda\partial a_2 + \Lambda a_{12},\\
&u_9^{\Lambda-1}a_4 = a_{14} + 2\partial a_4, \quad u_{10}^{\Lambda-1}a_4 = (\Lambda-1)a_{14}, \quad u_{11}^{\Lambda-1}a_4 = 0, \quad u_{12}^{\Lambda-1}a_4 = 2\Lambda\partial a_6,\\
&u_{13}^{\Lambda-1}a_4 = -\Lambda a_{16} + 2\partial a_{10}, \quad u_{14}^{\Lambda-1}a_4 = 0, \quad u_{15}^{\Lambda-1}a_4 = -2\partial a_{11}, \quad u_{16}^{\Lambda-1}a_4 = 2\partial a_{14}.
\end{aligned}$$

Similarly, the following is a set of $\mathbb{C}[\partial]$-generators for $N_5^{E_0}$.

$$\begin{aligned}
&u_1^{\Lambda-1}a_5 = a_5, \quad u_2^{\Lambda-1}a_5 = a_{10}, \quad u_3^{\Lambda-1}a_5 = -a_7, \quad u_4^{\Lambda-1}a_5 = a_{11}, \quad u_5^{\Lambda-1}a_5 = 0,\\
&u_6^{\Lambda-1}a_5 = a_{14}, \quad u_7^{\Lambda-1}a_5 = 0, \quad u_8^{\Lambda-1}a_5 = -\Lambda a_{13}, \quad u_9^{\Lambda-1}a_5 = a_{15}, \quad u_{10}^{\Lambda-1}a_5 = 0,\\
&u_{11}^{\Lambda-1}a_5 = 0, \quad u_{12}^{\Lambda-1}a_5 = -a_{16}, \quad u_{13}^{\Lambda-1}a_5 = 0, \quad u_{14}^{\Lambda-1}a_5 = 0, \quad u_{15}^{\Lambda-1}a_5 = 0,\\
&u_{16}^{\Lambda-1}a_5 = 0.
\end{aligned}$$

Therefore

$$S^\Lambda = \{a_4, a_5, a_6, a_7, a_9 - 2\partial a_1, a_{10}, a_{11}, a_{12} + 2\partial a_2, a_{13}, a_{14}, a_{15}, a_{16}\}$$

is a set of $\mathbb{C}[\partial]$-generators for N^{E_0}, which implies that $\{a_1, a_2, a_3, a_8\}$ is a $\mathbb{C}[\partial]$-basis for $(M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N)^{E_0}$ in the case when $\Lambda \geq 3$.

In the case when $\Lambda = 2$ the set $\{u_i^{\Lambda-1}a_4, u_i^{\Lambda-1}a_5 | 1 \leq i \leq 16, i \neq 11\}$ generate N^{E_0}. But $u_{11}^{\Lambda-1}a_4 = u_{11}^{\Lambda-1}a_5 = 0$, and hence $\{a_1, a_2, a_3, a_8\}$ is also a $\mathbb{C}[\partial]$-basis for $(M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N)^{E_0}$ in this case as well.

In the case when $\Lambda = 1$ we note that $a_{11} = 0$ and $\{u_i^{\Lambda-1}a_4, u_i^{\Lambda-1}a_5 | 1 \leq i \leq 16, i \neq 4, 5, 10, 11, 14, 15\}$ generates N^{E_0} over $\mathbb{C}[\partial]$. From the formulas above one sees that a set of $\mathbb{C}[\partial]$-generators for N^{E_0} is given by the set S^Λ above, but with a_{11} removed. Hence the quotient module is again generated freely over $\mathbb{C}[\partial]$ by $\{a_1, a_2, a_3, a_8\}$.

Hence in the case when $\Lambda \geq 1$ the quotient module $(M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N)^{E_0}$ is generated freely over $\mathbb{C}[\partial]$ by $\{a_1, a_2, a_3, a_8\}$. Now a_1 has H_0-weight Λ, a_2 and a_3 both have H_0-weight $\Lambda + 1$, and a_8 has H_0-weight $\Lambda + 2$. Therefore $M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N$ has rank $(\Lambda+1) + 2(\Lambda+2) + (\Lambda+3) = 4\Lambda + 8$ over $\mathbb{C}[\partial]$. So it remains to show that $M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N$ is irreducible.

We again study $M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N$ as a $\mathfrak{V}_+ \oplus sl_2$-module. It is easy to check that L_n, $n \geq 1$, annihilates a_1, a_2, a_3, a_8 and hence $M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2},\Lambda)/N$ is a direct sum of the following four irreducible $\mathfrak{V}_+ \oplus sl_2$-modules: $\mathbb{C}[\partial]V_1 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+2}{2}) \boxtimes$

U^Λ, $\mathbb{C}[\partial]V_2 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda+1}$, $\mathbb{C}[\partial]V_3 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda+1}$ and $\mathbb{C}[\partial]V_8 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda}{2})\boxtimes U^{\Lambda+2}$, where V_i is the sl_2-submodule generated by the vector a_i. Again $\mathbb{C}[\partial]V_2 \cong \mathbb{C}[\partial]V_3$ as $\mathfrak{V}_+ \oplus sl_2$-modules. Now we compute

$$G^{--}_{\frac{1}{2}}(\alpha a_2 + \beta a_3) = 2\alpha(\Lambda+1)a_1, \quad \forall \alpha, \beta \in \mathbb{C},$$
$$G^{-+}_{\frac{1}{2}} a_3 = -2(\Lambda+1)a_1, \quad F_1 a_8 = -2(\Lambda+1)a_1.$$

Therefore $M_{\mathfrak{N}^4_+}(-\frac{\Lambda+2}{2}, \Lambda)/N$ is irreducible.

Now consider the case of $\Lambda = 0$. By Proposition 6.1 a_6, a_7 and $a_9 - 2\partial a_1$ are singular vectors inside $M_{\mathfrak{N}^4_+}(-1,0)$. Let N_6, N_7 and N_9 be the $\mathfrak{g}$-submodules generated by a_6, a_7 and $a_9 - 2\partial a_1$, respectively, and put $N = N_6 + N_7 + N_9$. We note that a_6, a_7 and $a_9 - 2\partial a_1$ have H_0-weight 0, hence $N_6^{E_0}$ is generated over $\mathbb{C}[\partial]$ by $\{u_i^\Lambda a_6 | 1 \le i \le 16, i \ne 4,5,10,11,14,15\}$ and similarly for $N_7^{E_0}$ and $N_9^{E_0}$. We first compute a set of $\mathbb{C}[\partial]$-generators for $N_6^{E_0}$.

$$u_1^\Lambda a_6 = a_6, \quad u_2^\Lambda a_6 = 0, \quad u_3^\Lambda a_6 = 2\partial a_2 + a_{12}, \quad u_6^\Lambda a_6 = 0,$$
$$u_7^\Lambda a_6 = 4\partial^2 a_1 - 2\partial a_9 + a_{16}, \quad u_8^\Lambda a_6 = 0, \quad u_9^\Lambda a_6 = 4\partial a_6,$$
$$u_{12}^\Lambda a_6 = 0, \quad u_{13}^\Lambda a_6 = 4\partial^2 a_2 + 2\partial a_{12}, \quad u_{16}^\Lambda a_6 = 4\partial^2 a_6.$$

A set of $\mathbb{C}[\partial]$-generators for $N_7^{E_0}$ is given as follows.

$$u_1^\Lambda a_7 = a_7, \quad u_2^\Lambda a_7 = a_{13}, \quad u_3^\Lambda a_7 = 0, \quad u_6^\Lambda a_7 = a_{16}, \quad u_7^\Lambda a_7 = 0,$$
$$u_8^\Lambda a_7 = 0, \quad u_9^\Lambda a_7 = 0, \quad u_{12}^\Lambda a_7 = 0, \quad u_{13}^\Lambda a_7 = 0, \quad u_{16}^\Lambda a_7 = 0.$$

Finally we have the following set of $\mathbb{C}[\partial]$-generators for $N_9^{E_0}$.

$$u_1^\Lambda(a_9 - 2\partial a_1) = a_9 - 2\partial a_1, \quad u_2^\Lambda(a_9 - 2\partial a_1) = -a_{12} - 2\partial a_2,$$
$$u_3^\Lambda(a_9 - 2\partial a_1) = a_{13}, \quad u_6^\Lambda(a_9 - 2\partial a_1) = -2\partial a_6, \quad u_7^\Lambda(a_9 - 2\partial a_1) = 2\partial a_7,$$
$$u_8^\Lambda(a_9 - 2\partial a_1) = 0, \quad u_9^\Lambda(a_9 - 2\partial a_1) = 2a_{16}, \quad u_{12}^\Lambda(a_9 - 2\partial a_1) = 0,$$
$$u_{13}^\Lambda(a_9 - 2\partial a_1) = 2\partial a_{13}, \quad u_{16}^\Lambda(a_9 - 2\partial a_1) = 2\partial a_{16}.$$

From this it follows that $\{a_6, a_7, a_9 - 2\partial a_1, a_{12} + 2\partial a_2, a_{13}, a_{16}\}$ generate N^{E_0} over $\mathbb{C}[\partial]$. But $a_4 = a_5 = a_{10} = a_{11} = a_{14} = a_{15} = 0$, and thus $(M_{\mathfrak{N}^4_+}(-1,0)/N)^{E_0}$ is generated over $\mathbb{C}[\partial]$ by the vectors a_1, a_2, a_3 and a_8, which takes us back to the case when $\Lambda \ge 1$, except that here $\mathbb{C}[\partial]V_8$ is not irreducible. It contains a unique irreducible submodule isomorphic to $L_{\mathfrak{V}_+}(1) \boxtimes U^2$ generated by ∂a_8. But then the above calculation plus the fact that

$$F_2 \partial a_8 = -4(\Lambda+1)a_1$$

show that $M_{\mathfrak{N}^4_+}(-1,0)/N$ is irreducible of rank 8. $\square$

Theorem 6.1. *The modules $L_{\mathfrak{N}^4_+}(\Delta, \Lambda)$, for $\Delta \in \mathbb{C}$ and $\Lambda \in \mathbb{Z}_+$, form a complete list of non-isomorphic finite (over $\mathbb{C}[\partial]$) irreducible $SK(1,4)_+$-modules. Furthermore $L_{\mathfrak{N}^4_+}(\Delta, \Lambda)$ as a $\mathbb{C}[\partial]$-module has rank*

i. 4Λ, *in the case* $2\Delta - \Lambda = 0$,
ii. $4\Lambda + 8$, *in the case* $2\Delta + \Lambda + 2 = 0$.
iii. $16\Lambda + 16$, *in all other cases.*

Furthermore the $\mathbb{C}[\partial]$-rank of $L_{\mathfrak{N}^4_+}(\Delta, \Lambda)_{\bar{0}}$ equals the $\mathbb{C}[\partial]$-rank of $L_{\mathfrak{N}^4_+}(\Delta, \Lambda)_{\bar{1}}$ in all cases.

Remark 6.2. Translating the above theorem into the languages of modules over conformal algebras and of conformal modules is again straightforward. We therefore obtain that all finite irreducible modules over the "small" $N = 4$ conformal superalgebra are of the form $L_{\mathfrak{N}^4}(\alpha, \Delta, \Lambda)$, where $\alpha, \Delta \in \mathbb{C}$ and $\Lambda \in \mathbb{Z}_+$. The definition of these modules and also the action of the conformal superalgebra on them are easily gotten from our explicit description of a $\mathbb{C}[\partial]$-basis in this section and hence omitted, as to reproduce them would take up quite a significant portion of space. Again we only note that the adjoint module is isomorphic to $L_{\mathfrak{N}^4}(0, 1, 2)$.

7. Finite irreducible Modules over the "big" $N = 4$ conformal superalgebra

In this section we give a classification of finite irreducible conformal modules over the contact superalgebra $K(1,4)$, also known as the "big" $N = 4$ superconformal algebra. Our approach is based on our results obtained in Section 6.

Recall from Section 6 that L^β_n, X^β_n and x^β_r, where $X = H, E, F$, $x = G^{++}$, G^{-+},G^{+-},G^{--}, $n \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$ and the fixed number β is either 1 or -1, provide a basis for a copy of $SK(1,4)$ inside $K(1,4)$. In this section it will be convenient to distinguish these two copies. We therefore denote the copy obtained by setting $\beta = 1$ simply by $SK(1,4)$, while the copy obtained by setting $\beta = -1$ by $\overline{SK}(1,4)$. It is easy to see from our formulas that $K(1,4) = SK(1,4) + \overline{SK}(1,4)$. Similarly we distinguish the basis elements of $SK(1,4)$ and $\overline{SK}(1,4)$ as follows. The generators inside $SK(1,4)$ will be denoted by L_n, X_n, x_r, while generators inside $\overline{SK}(1,4)$ will be denoted by $\overline{L}_n, \overline{X}_n, \overline{x}_r$, where again $X = H, E, F$, $x = G^{++}$,G^{-+},G^{+-},G^{--}. Of course we have $x_{-\frac{1}{2}} = \overline{x}_{-\frac{1}{2}}$, $L_{-1} = \overline{L}_{-1}$ and $L_0 = \overline{L}_0$.

Remark 7.1. The map $\phi : SK(1,4) \to \overline{SK}(1,4)$ defined by $\phi(L_n) = \overline{L}_n$, $\phi(X_n) = \overline{X}_n$, $\phi(G^{++}_r) = \overline{G}^{++}_r$, $\phi(G^{+-}_r) = \overline{G}^{-+}_r$, $\phi(G^{-+}_r) = \overline{G}^{+-}_r$ and $\phi(G^{--}_r) = \overline{G}^{--}_r$, where $n \in \mathbb{Z}$ and $r \in \frac{1}{2} + \mathbb{Z}$ is an isomorphism of Lie superalgebras. Thus all formulas in Section 6 with $\phi(L_n)$, $\phi(X_n)$ and $\phi(x_r)$ replacing L_n, X_n and x_r, respectively, remain valid.

Let $\mathfrak{g} = K(1,4)_+$ be the annihilation subalgebra of $K(1,4)$ so that we have $\mathfrak{g} = SK(1,4)_+ + \overline{SK}(1,4)_+$, the sum of the corresponding annihilation subalgebras. We have as before $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$, where $j \in \frac{1}{2} + \mathbb{Z}$. Furthermore $\mathfrak{g}_- = SK(1,4)_- = \overline{SK}(1,4)_-$ and $\mathfrak{g}_0 = \mathbb{C}L_0 \oplus sl_2 \oplus \overline{sl}_2 \cong cso_4$, where sl_2 and $\overline{sl}_2$ denote two copies of the Lie algebra sl_2, generated by H_0, E_0, F_0 and $\overline{H}_0, \overline{E}_0, \overline{F}_0$, respectively.

Let $U^{\Delta,\Lambda,\overline{\Lambda}}$ be the finite-dimensional irreducible $sl_2 \oplus \overline{sl}_2$-module of highest weight $(\Lambda, \overline{\Lambda}) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ on which L_0 acts as the scalar $\Delta \in \mathbb{C}$ and let $v_{\Delta,\Lambda,\overline{\Lambda}}$ denote a highest weight vector in $U^{\Delta,\Lambda,\overline{\Lambda}}$ so that $H_0 v_{\Delta,\Lambda,\overline{\Lambda}} = \Lambda v_{\Delta,\Lambda,\overline{\Lambda}}$, $\overline{H}_0 v_{\Delta,\Lambda,\overline{\Lambda}} = \overline{\Lambda} v_{\Delta,\Lambda,\overline{\Lambda}}$ and $L_0 v_{\Delta,\Lambda,\overline{\Lambda}} = \Delta v_{\Delta,\Lambda,\overline{\Lambda}}$. Regarding $U^{\Delta,\Lambda,\overline{\Lambda}}$ as a module over $\mathfrak{L}_0 = \bigoplus_{j \geq 0} \mathfrak{g}_j$ it follows from Theorem 3.1 that every finite irreducible $\mathfrak{g}$-module is a quotient of $M_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda}) = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{L}_0} U^{\Delta,\Lambda,\overline{\Lambda}}$. The unique irreducible quotient will be denoted by $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})$.

Now $M_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})$ is a completely reducible $\mathfrak{g}_0$-module, and the subspace of $\mathbb{C}E_0 \oplus \mathbb{C}\overline{E}_0$-invariants, denoted by $M_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})^{E_0, \overline{E}_0}$, is a free $\mathbb{C}[\partial]$-submodule of $M_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})$. We write down explicit formulas for a $\mathbb{C}[\partial]$-basis for the space $M_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})^{E_0, \overline{E}_0}$, which in the case when $\Lambda, \overline{\Lambda} \geq 2$ is as follows:

$$b_1 = v_{\Delta,\Lambda,\overline{\Lambda}}, \quad b_2 = G^{++}_{-\frac{1}{2}} v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_3 = (\Lambda G^{-+}_{-\frac{1}{2}} - G^{++}_{-\frac{1}{2}} F_0) v_{\Delta,\Lambda,\overline{\Lambda}}, \quad b_4 = (\overline{\Lambda} G^{+-}_{-\frac{1}{2}} - G^{++}_{-\frac{1}{2}} \overline{F}_0) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_5 = (\Lambda\overline{\Lambda} G^{--}_{-\frac{1}{2}} - \overline{\Lambda} G^{+-}_{-\frac{1}{2}} F_0 - \Lambda G^{-+}_{-\frac{1}{2}} \overline{F}_0 + G^{++}_{-\frac{1}{2}} F_0 \overline{F}_0) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_6 = G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} v_{\Delta,\Lambda,\overline{\Lambda}}, \quad b_7 = \big(\Lambda (G^{-+}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} + G^{++}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}}) - 2 G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} F_0\big) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_8 = ((\Lambda - 1)(-\Lambda G^{-+}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} + G^{-+}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} F_0 + G^{++}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} F_0) - G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} F_0^2) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_9 = G^{++}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} v_{\Delta,\Lambda,\overline{\Lambda}}, \quad b_{10} = \big(\overline{\Lambda} (G^{++}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} - G^{-+}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}}) - 2 G^{++}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} \overline{F}_0\big) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_{11} = \big((\overline{\Lambda} - 1)(-\overline{\Lambda} G^{+-}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} + G^{+-}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} \overline{F}_0 + G^{++}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} \overline{F}_0) - G^{++}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} \overline{F}_0^2\big) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_{12} = G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} v_{\Delta,\Lambda,\overline{\Lambda}}, \quad b_{13} = (\Lambda G^{++}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} - G^{++}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}}) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_{14} = (\overline{\Lambda} G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} - G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}}) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_{15} = \big(\Lambda G^{-+}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} (\overline{\Lambda} G^{+-}_{-\frac{1}{2}} - G^{++}_{-\frac{1}{2}} \overline{F}_0) + \overline{\Lambda} G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} (G^{--}_{-\frac{1}{2}} F_0 - G^{-+} F_0 \overline{F}_0)\big) v_{\Delta,\Lambda,\overline{\Lambda}},$$
$$b_{16} = \big(G^{++}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} G^{-+}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}} - \partial (G^{-+}_{-\frac{1}{2}} G^{+-}_{-\frac{1}{2}} + G^{++}_{-\frac{1}{2}} G^{--}_{-\frac{1}{2}})\big) v_{\Delta,\Lambda,\overline{\Lambda}}.$$

In the case when $\Lambda = \overline{\Lambda} = 1$ (respectively $\Lambda = \overline{\Lambda} = 0$) we have $b_8 = b_{11} = 0$ (respectively $b_3 = b_4 = b_5 = b_7 = b_8 = b_{10} = b_{11} = b_{13} = b_{14} = b_{15} = 0$), thus giving us 14 (respectively 6) generators. Other cases are easily described as well, however, we will not need them because of Proposition 7.1 below. Thus we will omit them.

We will, as before, denote the coefficient of $v_{\Delta,\Lambda,\overline{\Lambda}}$ in b_i by $u_i^{\Lambda,\overline{\Lambda}}$ for $1 \leq i \leq 16$. In the case when $\Lambda = \overline{\Lambda}$, which is the only case we will be concerned with in what follows, we simply write u_i^{Λ} for $u_i^{\Lambda,\Lambda}$ and also $v_{\Delta,\Lambda}$ for $v_{\Delta,\Lambda,\Lambda}$.

Proposition 7.1. *If $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda})$ is a reducible $\mathfrak{g}$-module, then either $2\Delta - \Lambda = 2\Delta - \overline{\Lambda} = 0$ or else $2\Delta + \Lambda + 2 = 2\Delta + \overline{\Lambda} + 2 = 0$. In particular if $\Lambda \neq \overline{\Lambda}$, then $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda})$ is irreducible.*

Proof. As a module over $SK(1,4)_+$ we have $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda}) = U(\mathfrak{g}_-) \otimes U^{\Delta,\Lambda,\overline{\Lambda}}$ is a direct sum of $\overline{\Lambda}+1$ copies of $M_{\mathfrak{N}_+^4}(\Delta, \Lambda)$, generated by the highest weight vectors $\overline{F}_0^j v_{\Delta,\Lambda,\overline{\Lambda}}$, where $0 \leq j \leq \overline{\Lambda}$. Since the $\overline{H}_0$-weights of the $\overline{F}_0^j v_{\Delta,\Lambda,\overline{\Lambda}}$'s are all distinct for distinct j's it follows that these modules as $SK(1,4)_+ \rtimes \mathbb{C}\overline{H}_0$-modules are all non-isomorphic. Therefore if $M_{\mathfrak{N}_+^4}(\Delta, \Lambda)$ is irreducible over $SK(1,4)_+$, then $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda})$ is irreducible over $\mathfrak{g}$. From this and Corollary 6.1 we thus conclude that in the case when $\Delta - 2\Lambda \neq 0$ and $\Delta + 2\Lambda + 2 \neq 0$ the $\mathfrak{g}$-module $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda})$ is irreducible.

By symmetry we conclude that if $\Delta - 2\overline{\Lambda} \neq 0$ and $\Delta + 2\overline{\Lambda} + 2 \neq 0$, then $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda})$ is irreducible over $\mathfrak{g}$ as well.

Therefore $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \overline{\Lambda})$ is possibly reducible only if both Λ and $\overline{\Lambda}$ satisfy one of the two linear equations $\Delta - 2x = 0$ and $\Delta + 2x + 2 = 0$. But the case $\Delta - 2\Lambda = 0$ and $\Delta + 2\overline{\Lambda} + 2 = 0$ is not possible, since both Λ and $\overline{\Lambda}$ are non-negative integers. By the same token $\Delta - 2\overline{\Lambda} = 0$ and $\Delta + 2\Lambda + 2 = 0$ is not possible, either. Hence either we have $\Delta - 2\Lambda = 0$ and $\Delta - 2\overline{\Lambda} = 0$ or else $\Delta + 2\Lambda + 2 = 0$ and $\Delta + 2\overline{\Lambda} + 2 = 0$. In either case we must have $\Lambda = \overline{\Lambda}$. □

The next step is to analyze proper singular vectors inside $M_{\mathfrak{S}^4}(\Delta, \Lambda, \overline{\Lambda})$. (The definitions of singular vectors and proper singular vectors of $\mathfrak{g}$ are of course analogous.) By Proposition 7.1 proper singular vectors exist only if $\Lambda = \overline{\Lambda}$ with either $2\Delta + \Lambda = 0$ or $2\Delta + \Lambda + 2 = 0$.

Proposition 7.2. *A complete list of proper singular vectors inside $M_{\mathfrak{S}_+^4}(\Delta, \Lambda, \Lambda)$ is given by:*

i. αb_2, $\alpha \neq 0$, *in the case* $2\Delta - \Lambda = 0$.
ii. αb_5, $\alpha \neq 0$, *in the case* $2\Delta + \Lambda + 2 = 0$ *and* $\Lambda \geq 1$.

Proof. Since as a $SK(1,4)_+$-module $M_{\mathfrak{S}^4}(\Delta, \Lambda, \Lambda)$ is a direct sum of $\Lambda + 1$ copies of $M_{\mathfrak{N}^4}(\Delta, \Lambda)$ we obtain a description of the vector space spanned by all proper $SK(1,4)_+$-singular vectors by virtue of Proposition 6.1. But as a $\overline{SK}(1,4)_+$-module $M_{\mathfrak{S}^4}(\Delta, \Lambda, \Lambda)$ is also a direct sum of $\Lambda + 1$ copies of $M_{\mathfrak{N}^4}(\Delta, \Lambda)$, from which we obtain similarly a description of the vector space spanned by all proper $\overline{SK}(1,4)_+$-singular vectors (see Remark 7.1). The intersection of these two spaces is the space of proper singular vectors.

In the case when $2\Delta - \Lambda = 0$ it follows from Proposition 6.1 that the space of proper $SK(1,4)_+$-singular vectors is spanned by $G^{++}_{-\frac{1}{2}}\overline{F}^j_0 v_{\Delta,\Lambda}$, $G^{+-}_{-\frac{1}{2}}\overline{F}^j_0 v_{\Delta,\Lambda}$ and $G^{++}_{-\frac{1}{2}}G^{+-}_{-\frac{1}{2}}\overline{F}^j_0 v_{\Delta,\Lambda}$, for $0 \le j \le \Lambda$. On the other hand the space of proper $\overline{SK}(1,4)_+$-singular vectors is spanned by $G^{++}_{-\frac{1}{2}}F^j_0 v_{\Delta,\Lambda}$, $G^{-+}_{-\frac{1}{2}}F^j_0 v_{\Delta,\Lambda}$ and $G^{++}_{-\frac{1}{2}}G^{-+}_{-\frac{1}{2}}F^j_0 v_{\Delta,\Lambda}$, for $0 \le j \le \Lambda$. It is not hard to see that the intersection of these two spaces is the one-dimensional space spanned by $G^{++}_{-\frac{1}{2}}v_{\Delta,\Lambda}$, which is b_2.

Other cases are analogous and so we omit the details. $\square$

Proposition 7.3. *Suppose that* $2\Delta - \Lambda = 0$. *Then* $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \Lambda)$ *is a free* $\mathbb{C}[\partial]$-*module of rank* $8\Lambda(\Lambda+1)$.

Proof. By Proposition 7.2 b_2 is a singular vector in $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)$. Consider N, the $\mathfrak{g}$-submodule generated by b_2. Then we have $N = U(\mathfrak{g}_-)V$, where V is the irreducible $sl_2 \oplus \overline{sl}_2$-submodule generated by b_2. Let us compute the space $N^{E_0,\overline{E}_0}$, the space of $(\mathbb{C}E_0 \oplus \mathbb{C}\overline{E}_0)$-invariants inside N. Since the $(H_0, \overline{H}_0)$-weight of b_2 is $(\Lambda+1, \Lambda+1)$, we know that $N^{E_0,\overline{E}_0}$ is a free $\mathbb{C}[\partial]$-module generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda+1}b_2 | 1 \le i \le 16\}$. We have

$$u_1^{\Lambda+1}b_2 = b_2, \quad u_2^{\Lambda+1}b_2 = 0, \quad u_3^{\Lambda+1}b_2 = -(\Lambda+2)b_9, \quad u_4^{\Lambda+1}b_2 = -(\Lambda+2)b_6,$$
$$u_5^{\Lambda+1}b_2 = -\frac{\Lambda+2}{2}(b_7 + b_{10} + 4(\Lambda+1)\partial b_1), \quad u_6^{\Lambda+1}b_2 = 0, \quad u_7^{\Lambda+1}b_2 = -(\Lambda+3)b_{12},$$
$$u_8^{\Lambda+1}b_2 = -(\Lambda+2)(b_{13} - 2\partial b_3), \quad u_9^{\Lambda+1}b_2 = 0, \quad u_{10}^{\Lambda+1}b_2 = (\Lambda+3)b_{12} - 4(\Lambda+1)\partial b_2,$$
$$u_{11}^{\Lambda+1}b_2 = -(\Lambda+2)(b_{14} - 2\partial b_4), \quad u_{12}^{\Lambda+1}b_2 = 0, \quad u_{13}^{\Lambda+1}b_2 = -2(\Lambda+2)\partial b_9,$$
$$u_{14}^{\Lambda+1}b_2 = -2(\Lambda+2)\partial b_6, \quad u_{15}^{\Lambda+1}b_2 = -4(\Lambda+1)\partial^2 b_1 - (\Lambda+2)^2 b_{16} + (\Lambda+2)\partial b_7,$$
$$u_{16}^{\Lambda+1}b_2 = -4\partial b_{12}.$$

It follows that $N^{E_0,\overline{E}_0}$ is generated over $\mathbb{C}[\partial]$ by the set

$$S^\Lambda = \{b_2, b_6, b_7 + b_{10} + 4(\Lambda+2)\partial b_1, b_9, b_{12}, b_{13} - 2\partial b_3, b_{14} - 2\partial b_4,$$
$$b_{16} - (\frac{1}{\Lambda+2})\partial b_7 - 2\frac{(\Lambda+1)}{(\Lambda+2)^2}\partial^2 b_1\}.$$

In the case when $\Lambda \ge 2$ it follows from the description of S^Λ that $\{b_1, b_3, b_4, b_5, b_8, b_{10} + 2\Lambda\partial b_1, b_{11}, b_{15}\}$ is a $\mathbb{C}[\partial]$-basis for the $(\mathbb{C}E_0 \oplus \mathbb{C}\overline{E}_0)$-invariants of the quotient space $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$. (The choice of $b_{10} + 2\Lambda\partial b_1$ instead of just b_{10} will be explained later.)

The $(L_0, H_0, \overline{H}_0)$-weights of b_1, b_3, b_4, b_5, b_8, $b_{10} + 2\Lambda\partial b_1$, b_{11}, b_{15} are $(\Delta, \Lambda, \Lambda)$, $(\Delta + \frac{1}{2}, \Lambda-1, \Lambda+1)$, $(\Delta + \frac{1}{2}, \Lambda+1, \Lambda-1)$, $(\Delta + \frac{1}{2}, \Lambda-1, \Lambda-1)$, $(\Delta+1, \Lambda-2, \Lambda)$, $(\Delta+1, \Lambda, \Lambda)$, $(\Delta+1, \Lambda, \Lambda-2)$, $(\Delta + \frac{3}{2}, \Lambda-1, \Lambda-1)$, respectively. Hence

$M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ is a free $\mathbb{C}[\partial]$-module of rank $8\Lambda(\Lambda+1)$. So we need to show that $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ is irreducible.

Now L_n, $n \geq -1$, together with E_0, H_0, F_0 and $\overline{E}_0, \overline{H}_0, \overline{F}_0$ generate a copy of $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$, which thus allow us to study the $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-module structure of $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$. We can easily check that L_n, for $n \geq 1$, annihilates the vectors b_1, b_3, b_4, b_5, b_8, $b_{10} + 2\Lambda\partial b_1$, b_{11}, b_{15}. (We want to point out that b_{10} is not annihilated by L_n, for $n \geq 1$, hence the choice of $b_{10} + 2\Lambda\partial b_1$.) Thus $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ as a $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-module is a direct sum of the following eight irreducible modules: $\mathbb{C}[\partial]V_1 \cong L_{\mathfrak{V}_+}(\frac{\Lambda}{2}) \boxtimes U^{\Lambda,\Lambda}$, $\mathbb{C}[\partial]V_3 \cong L_{\mathfrak{V}_+}(\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda-1,\Lambda+1}$, $\mathbb{C}[\partial]V_4 \cong L_{\mathfrak{V}_+}(\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda+1,\Lambda-1}$, $\mathbb{C}[\partial]V_5 \cong L_{\mathfrak{V}_+}(\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda-1,\Lambda-1}$, $\mathbb{C}[\partial]V_8 \cong L_{\mathfrak{V}_+}(\frac{\Lambda+2}{2}) \boxtimes U^{\Lambda-2,\Lambda}$, $\mathbb{C}[\partial]V_{10} \cong L_{\mathfrak{V}_+}(\frac{\Lambda+2}{2}) \boxtimes U^{\Lambda,\Lambda}$, $\mathbb{C}[\partial]V_{11} \cong L_{\mathfrak{V}_+}(\frac{\Lambda+2}{2}) \boxtimes U^{\Lambda,\Lambda-2}$, $\mathbb{C}[\partial]V_{15} \cong L_{\mathfrak{V}_+}(\frac{\Lambda+3}{2}) \boxtimes U^{\Lambda-1,\Lambda-1}$, where V_i is the irreducible $sl_2 \oplus \overline{sl}_2$-module generated by b_i, for $i \neq 10$, and V_{10} is generated by $b_{10} + 2\Lambda\partial b_1$, and finally $U^{\mu,\mu'}$ denotes the irreducible $sl_2 \oplus \overline{sl}_2$-module of highest weight (μ, μ'). Note that as $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-modules they are all non-isomorphic and thus to show that $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ is irreducible, it suffices to show that one may send a $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-highest weight vector in any irreducible $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-component to the irreducible component containing the $\mathfrak{g}$-highest weight vectors. This follows from the following computation.

$$
\begin{aligned}
&G^{--}_{\frac{1}{2}} b_3 = 2(\Lambda+1)F_0 b_1, \quad \overline{G}^{--}_{\frac{1}{2}} b_4 = 2(\Lambda+1)\overline{F}_0 b_1,\\
&G^{++}_{\frac{1}{2}} b_5 = -2\Lambda^2(\Lambda+1)b_1, \quad E_1 b_8 = 2\Lambda(\Lambda-1)(\Lambda+1)b_1,\\
&\overline{F}_1(b_{10} + 2\Lambda\partial b_1) = -2(\Lambda+2)\overline{F}_0 b_1, \quad \overline{E}_1 b_{11} = 2\Lambda(\Lambda-1)(\Lambda+1)b_1,\\
&\overline{G}^{++}_{\frac{3}{2}} b_{15} = -2\Lambda^2(\Lambda+1)b_1.
\end{aligned}
$$

Now if $\Lambda = 1$ the vectors $b_8 = b_{11} = 0$. Therefore $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ is $\mathbb{C}[\partial]V_1 \oplus \mathbb{C}[\partial]V_3 \oplus \mathbb{C}[\partial]V_4 \oplus \mathbb{C}[\partial]V_5 \oplus \mathbb{C}[\partial]V_{10} \oplus \mathbb{C}[\partial]V_{15}$. But then the above calculation also shows that $M_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ is irreducible. The rank of $L_{\mathfrak{S}^4_+}(\frac{\Lambda}{2}, \Lambda, \Lambda)$ is then $4+3+3+1+4+1 = 16$, which is equal to $8\Lambda(\Lambda+1)$ in the case when $\Lambda = 1$.

Finally when $\Lambda = 0$, the vectors $b_3 = b_4 = b_5 = b_7 = b_8 = b_{10} = b_{11} = b_{13} = b_{14} = b_{15} = 0$ and S^Λ reduces to $\{b_2, b_6, \partial b_1, b_9, b_{12}, b_{16}\}$. Hence $M_{\mathfrak{S}^4_+}(0,0,0)/N = \mathbb{C}b_1$ is the trivial module and so has rank 0. □

Proposition 7.4. *Suppose that* $2\Delta + \Lambda + 2 = 0$ *and* $\Lambda \geq 1$. *Then* $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \Lambda)$ *is a free* $\mathbb{C}[\partial]$*-module of rank* $8(\Lambda+1)(\Lambda+2)$.

Proof. By Proposition 7.2 b_5 is a singular vector in $M_{\mathfrak{S}^4_+}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)$. Let N be the $\mathfrak{g}$-submodule generated by b_5 so that $N = U(\mathfrak{g}_-)V$, where V is the irreducible $sl_2 \oplus \overline{sl}_2$-submodule generated by b_5. Consider $N^{E_0,\overline{E}_0}$, the subspace in N of

$\mathbb{C}E_0 \oplus \mathbb{C}\overline{E}_0$-invariants. Now the $(H_0, \overline{H}_0)$-weight of b_5 is $(\Lambda-1, \Lambda-1)$ and so $N^{E_0,\overline{E}_0}$ is a free $\mathbb{C}[\partial]$-module generated over $\mathbb{C}[\partial]$ by $\{u_i^{\Lambda-1}b_5 | 1 \le i \le 16\}$. We have

$$
\begin{aligned}
&u_1^{\Lambda-1}b_5 = b_5, \quad u_2^{\Lambda-1}b_5 = \frac{1}{2}(b_7 + b_{10}), \quad u_3^{\Lambda-1}b_5 = -\Lambda b_8, \quad u_4^{\Lambda-1}b_5 = -\Lambda b_{11},\\
&u_5^{\Lambda-1}b_5 = 0, \quad u_6^{\Lambda-1}b_5 = \Lambda b_{14}, \quad u_7^{\Lambda-1}b_5 = -(\Lambda-1)b_{15}, \quad u_8^{\Lambda-1}b_5 = 0,\\
&u_9^{\Lambda-1}b_5 = \Lambda b_{13}, \quad u_{10}^{\Lambda-1}b_5 = (\Lambda-1)b_{15}, \quad u_{11}^{\Lambda-1}b_5 = 0,\\
&u_{12}^{\Lambda-1}b_5 = \Lambda(\Lambda b_{16} + \partial b_7), \quad u_{13}^{\Lambda-1}b_5 = 0, \quad u_{14}^{\Lambda-1}b_5 = 0, \quad u_{15}^{\Lambda-1}b_5 = 0,\\
&u_{16}^{\Lambda-1}b_5 = \partial b_{15}.
\end{aligned}
$$

It follows that in the case $\Lambda \ge 2$ that $N^{E_0,\overline{E}_0}$ is generated over $\mathbb{C}[\partial]$ by the set

$$S^\Lambda = \{b_5, b_7 + b_{10}, b_8, b_{11}, b_{13}, b_{14}, b_{15}, \Lambda b_{16} + \partial b_7\}.$$

Hence in this case $\{b_1, b_2, b_3, b_4, b_6, b_9, b_{10} + 2\Lambda\partial b_1, b_{12}\}$ is a $\mathbb{C}[\partial]$-basis for the $(\mathbb{C}E_0 \oplus \mathbb{C}\overline{E}_0)$-invariants of $M_{\mathfrak{S}_+^4}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)/N$.

The $(L_0, H_0, \overline{H}_0)$-weights of b_1, b_2, b_3, b_4, b_6, b_9, $b_{10} + 2\Lambda\partial b_1$, b_{12} are $(\Delta, \Lambda, \Lambda)$, $(\Delta + \frac{1}{2}, \Lambda+1, \Lambda+1)$, $(\Delta + \frac{1}{2}, \Lambda-1, \Lambda+1)$, $(\Delta + \frac{1}{2}, \Lambda+1, \Lambda-1)$, $(\Delta + 1, \Lambda + 2, \Lambda)$, $(\Delta+1, \Lambda, \Lambda+2)$, $(\Delta+1, \Lambda, \Lambda)$, $(\Delta + \frac{3}{2}, \Lambda+1, \Lambda+1)$, respectively. Hence $M_{\mathfrak{S}_+^4}(-\frac{\Lambda+2}{2}, \Lambda)/N$ is a free $\mathbb{C}[\partial]$-module of rank $8(\Lambda+1)(\Lambda+2)$. So we need to show that $M_{\mathfrak{S}_+^4}(\frac{\Lambda}{2}, \Lambda, \Lambda)/N$ is irreducible.

Again we will study the $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-module structure of $M_{\mathfrak{N}_+^4}(\frac{\Lambda}{2}, \Lambda)/N$. We can check directly that L_n, for $n \ge 1$, annihilates the vectors b_1, b_2, b_3, b_4, b_6, b_9, $b_{10} + 2\Lambda\partial b_1$, b_{12}. Thus $M_{\mathfrak{S}_+^4}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)/N$ as a $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-module is a direct sum of the following eight irreducible modules: $\mathbb{C}[\partial]V_1 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+2}{2}) \boxtimes U^{\Lambda,\Lambda}$, $\mathbb{C}[\partial]V_2 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda+1,\Lambda+1}$, $\mathbb{C}[\partial]V_3 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda-1,\Lambda+1}$, $\mathbb{C}[\partial]V_4 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda+1}{2}) \boxtimes U^{\Lambda+1,\Lambda-1}$, $\mathbb{C}[\partial]V_6 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda}{2}) \boxtimes U^{\Lambda+2,\Lambda}$, $\mathbb{C}[\partial]V_9 \cong L_{\mathfrak{V}_+}(-\frac{\Lambda}{2}) \boxtimes U^{\Lambda,\Lambda+2}$, $\mathbb{C}[\partial]V_{10} \cong L_{\mathfrak{V}_+}(-\frac{\Lambda}{2}) \boxtimes U^{\Lambda,\Lambda}$, $\mathbb{C}[\partial]V_{12} \cong L_{\mathfrak{V}_+}(-\frac{\Lambda-1}{2}) \boxtimes U^{\Lambda+1,\Lambda+1}$, where V_i is the irreducible $sl_2 \oplus \overline{sl}_2$-module generated by b_i, for $i \ne 10$, and V_{10} is generated by $b_{10} + 2\Lambda\partial b_1$, and $U^{\mu,\mu'}$ is the irreducible $sl_2 \oplus \overline{sl}_2$-module of highest weight (μ, μ'). Note these modules are all irreducible. Note further that they are all non-isomorphic. So as before to show that $M_{\mathfrak{S}_+^4}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)/N$ is irreducible, it suffices to show that one may send a $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-highest weight vector in any irreducible $(\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2)$-component to the irreducible component containing

the $\mathfrak{g}$-highest weight vectors. For this purpose we compute

$$\begin{aligned}
&G^{--}_{\frac{1}{2}} b_2 = 2(\Lambda+1)b_1, \quad \overline{G}^{+-}_{\frac{1}{2}} b_3 = -2\Lambda(\Lambda+1)b_1,\\
&G^{-+}_{\frac{1}{2}} b_4 = -2\Lambda(\Lambda+1)b_1, \quad F_1 b_6 = -2(\Lambda+1)b_1,\\
&\overline{F}_1 b_9 = -2(\Lambda+1)b_1 \quad \overline{F}_1(b_{10} + 2\Lambda\partial b_1) = 2\Lambda\overline{F}_0 b_1,\\
&\overline{G}^{--}_{\frac{3}{2}} b_{12} = 8(\Lambda+1)b_1.
\end{aligned}$$

This settles the case when $\Lambda \geq 2$.

In the case when $\Lambda = 1$ $N^{E_0,\overline{E}_0}$ is generated over $\mathbb{C}[\partial]$ by

$$S^\Lambda = \{b_5, b_7 + b_{10}, b_{13}, b_{14}, b_{16} + \partial b_7, \partial b_{15}\}.$$

Therefore $M_{\mathfrak{S}^4_+}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)/N$ contains a ∂-invariant (and hence $\mathfrak{g}$-invariant) vector b_{15}. Since in this case the vectors $b_8 = b_{11} = 0$, $M_{\mathfrak{S}^4_+}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)/(N + \mathbb{C}b_{15})$ as a $\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2$-module is isomorphic to $\mathbb{C}[\partial]V_1 \oplus \mathbb{C}[\partial]V_2 \oplus \mathbb{C}[\partial]V_3 \oplus \mathbb{C}[\partial]V_4 \oplus \mathbb{C}[\partial]V_6 \oplus \mathbb{C}[\partial]V_9 \oplus \mathbb{C}[\partial]V_{10} \oplus \mathbb{C}[\partial]V_{12}$. Every component is irreducible except for $\mathbb{C}[\partial]V_{12}$, which contains a unique (irreducible) $\mathfrak{V}_+ \oplus sl_2 \oplus \overline{sl}_2$-submodule isomorphic to $L_{\mathfrak{V}_+}(1) \otimes U^{2,2}$ generated by the highest weight vector ∂b_{15}. But then the above calculation plus the fact that

$$\overline{G}^{--}_{\frac{5}{2}} \partial b_{12} = 24(\Lambda+1)\partial b_1$$

also shows that $M_{\mathfrak{S}^4_+}(-\frac{\Lambda+2}{2}, \Lambda, \Lambda)/(N + \mathbb{C}b_{15})$ is irreducible. □

We summarize the results of this section in the following theorem.

Theorem 7.1. *The modules $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})$, for $\Delta \in \mathbb{C}$ and $\Lambda, \overline{\Lambda} \in \mathbb{Z}_+$, form a complete list of non-isomorphic finite (over $\mathbb{C}[\partial]$) irreducible $K(1,4)_+$-modules. Furthermore $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})$ as a $\mathbb{C}[\partial]$-module has rank*

i. $8\Lambda(\Lambda+1)$*, in the case* $2\Delta - \Lambda = 0$ *and* $\Lambda = \overline{\Lambda}$*,*
ii. $8(\Lambda+1)(\Lambda+2)$*, in the case* $2\Delta + \Lambda + 2 = 0$ *and* $\Lambda = \overline{\Lambda}$*.*
iii. $16(\Lambda+1)(\overline{\Lambda}+1)$*, in all other cases.*

Furthermore the $\mathbb{C}[\partial]$-rank of $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})_{\bar{0}}$ equals the $\mathbb{C}[\partial]$-rank of $L_{\mathfrak{S}^4_+}(\Delta, \Lambda, \overline{\Lambda})_{\bar{1}}$ in all cases.

Remark 7.2. Again the translation into the languages of modules over conformal algebras and of conformal modules is straightforward and hence is omitted. We thus obtain that all finite irreducible modules over the "big" $N = 4$ conformal superalgebra are of the form $L_{\mathfrak{S}^4}(\alpha, \Delta, \Lambda, \overline{\Lambda})$, where $\alpha, \Delta \in \mathbb{C}$ and $\Lambda, \overline{\Lambda} \in \mathbb{Z}_+$. Again the definition of these modules and the action of the conformal superalgebra on them are easily derived from our explicit description of a $\mathbb{C}[\partial]$-basis in this section. We note that the adjoint module is isomorphic to $M_{\mathfrak{S}^4}(0,0,0,0)$. This module is not simple, since $K(1,4)$ is not a simple Lie superalgebra. Its derived

algebra $K(1,4)'$ (which is a simple formal distribution Lie superalgebra) is an ideal in $K(1,4)$ of codimension 1 [11]. Thus the annihilation subalgebra of $K(1,4)'$ and $K(1,4)$ are identical, and hence their conformal modules are identical. Therefore the results in this section also give explicit description of irreducible conformal modules over $K(1,4)'$. We finally remark that the $K(1,4)'$ as a conformal module over $K(1,4)$ corresponds to $L_{\mathfrak{S}^4}(0, \frac{1}{2}, 1, 1)$.

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